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Abstract

The Four-Color Theorem (T4C) originated from the attempt to resolve the challenge of coloring maps over a plane or spherical surface. Over a century and a half, this problem underwent various abstractions until finding resolution in 1976. The proposed solution, which is groundbreaking, computationally calculates the multitude of representations of flat maps. Despite its resolution, the absence of a formal proof for this problem using traditional mathematical tools induces some unease, and a readily comprehensible solution is desirable. Nowadays, numerous articles propose diverse solutions to this problem but none of which have garnered official recognition. Taking a distinctive approach, solutions grounded in function properties and infinitesimal discretization were presented by this author in 2021 and 2023.

This article introduces an infinite covering of a map by a set of parallel lines, effectively capturing the structure of the map. A series of segmented straight lines are generated, and it is demonstrated that they are paintable with Four Colors, implying that the corresponding maps are colorable by Four Colors as well.

Keywords: Four-Color Theorem; Continuity of MAPs; line approximation coverture.

Introduction

The problem of the four $colors$ (1) arose in 1852 when Francis Guthrie observed that four $colors$ seemed to be sufficient to color the map of England. After a long process his conjecture was sent to Arthur Cayley, who published the problem in the Philosophical Magazine in 1879, and then the problem gained international notoriety. This problem was maintained open until 1977 when Kenneth Appel and Wolfgang Haken published the first widely accepted proof generating controversy due to the use of a computer-assisted proof in the demonstration. After then constant improvement was done using the same abordage. In 1997, Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas presented a simplified version of Haken's proof. In 2008, Georges Gonthier and Benjamin Werner published a proof using a computer-assisted proof system called Coq, formalizing Robertson et al.'s proof. In 2012, Thomas Hales announced a project called Flyspeck using another computer-assisted proof system called HOL Light. The project was completed in 2014 and published in the Forum of Mathematics Pi in 2017. In 2016, Marijn Heule, Oliver Kullmann, and Victor Marek presented a proof of the Four-Color Theorem using a resolution-based problem-solving method based on Boolean satisfiability (SAT). They reduced the problem to 633 Boolean formulas, each representing an unavoidable configuration, and used a SAT solver to verify their colorability that was published in the Journal of Automated Reasoning in 2017."

The purpose of this work is to return to the origin of the Four Color Problem and use old mathematical tools contained in good Calculus books (Arfken, Courant) to solve it. The concept underlying the preceding (Jansen 2021 and 2023) articles, as well as the current one, is to revisit the roots of the Four Color Theorem and employ classical mathematical tools found in reputable calculus texts (such as Arfken and Courant) to resolve it.

To depict a map on a surface (Planar Maps), the process involves distinguishing neighboring regions (countries, oceans, etc.) by painting them with different $colors$. Within a map, each region is characterized by three variables: the area of the region (always assumed to have a nonzero measure and to be more or less "well-behaved," avoiding peculiar shapes like fractals, etc.); a $label$ (the name of the region); and a color. The statement of the Four Color Theorem includes an explicit and necessary condition:

A1-) Neighboring regions must have different $colors$.

Additionally, there are implicit hypotheses:

A1'-) Neighboring regions must have different $label$;
B-) There is territorial continuity within each region, implying continuity in R2 inside a closed surface bounded by a contour. In simpler terms, within the same region, there is always a continuous path connecting two different points within the region;
C-) Each region has a single color and a single label. Consequently, a map can be considered a set of pairs $(color,\ label)$, and $color$ space and $label$l space can be defined as well;

D-) Non-neighboring regions can have the same color, but they must necessarily have different labels.

The Four-Colors Theorem aims to demonstrate the possibility of painting any map with four colors.

Approximating a MAP using a Straight Line. What happens if ?

To address this objective, some transformations are necessary. Given a unitary spherical surface $\psi\ (r = 1, \theta, \varphi) = 0$, in 2023 Jansen proposed to transform the frontiers of Regions $\chi_i(r = 1, \theta, \varphi) = 0$ into equations $\gamma_i(\theta, \varphi) = 0$ defined in a (θ, φ) plane where $\left(\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$ and $(0 \leq \varphi \leq 2\pi)$. Assuming that this transformation is completed, a dislocation of θ axis can be performed redefining the θ limits to $(0 \leq \theta < \pi)$ and with this second transformation the axes will be at origin $(0, 0)$. Now, the plane (θ, φ) containing their $\gamma_i(\theta, \varphi) = 0$ equations can accommodate (n-1) copies of itself along an extended φ axis as depicted in Fig 1. In this extended plane a linear one variable equation can be expressed as $f_n(\theta, \varphi) = \theta - \left(\frac{1}{(2n)}\right)\varphi = 0$. Thus $\theta = \left(\frac{1}{(2n)}\right)\varphi$ is the geometrical locus of a straight line equation drawn from point $(0, 0)$ to $(\pi, 2n\pi)$ as illustrated in Fig 1.

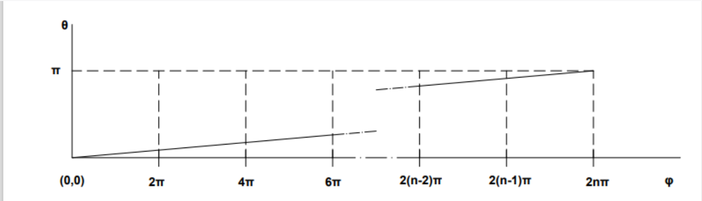

Fig 1: Schematic picture of (θ, φ) plane, its copies on an extended (θ, φ) plane (on extended φ axis) and the plot of the n-straight line. All $\gamma_i(\theta, \varphi) = 0$ are embedded in the copies of the (θ, φ) plane.

A second operation that can be done is the translation of all straight lines contained in $(2(k - 1)\pi \leq \varphi < 2k\pi)$to the first copy as in Fig 2.

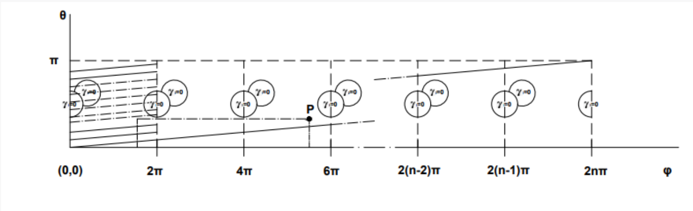

Fig 2: Depicts the coverture given by the l lines transported to the the first quadrant at the restrict φ interval $(0 \leq \varphi < 2\pi)$. Example of Regions delimited by $\gamma_1(\theta, \varphi) = 0$ and $\gamma_2(\theta, \varphi) = 0$ and their replication through the copies of (θ, φ) plane are presented

As can be inferred from Fig 2 (it is a self explained figure) it can be affirmed that as $n \to \infty$ a coverture of all Regions contained in the (θ, φ) plane is achieved and as $n \to \infty$ any point in the first copy correspond a point P at any desired ϵ distance to the n-straight line. All $\gamma_i(\theta, \varphi) = 0$ are intersected by the n-straight line and an order is established between the cutting points of the intersections of $\gamma_i(\theta, \varphi) = 0$ with the n-straight line $\theta - \left(\frac{1}{(2n)}\right)\varphi = 0$. A similar order was established in 2023 by Jansen in a discrete approximation of the same problem.

As shown in Fig 2, the l_{esime} line contained into the n-straight line between the $(2\pi l \leq \varphi < 2\pi(l + 1))$ limits may be transferred to the $(0 \leq \varphi < 2\pi)$ interval.

The pairs of cutting points $\left(\varphi_{initial_k}, \varphi_{final_k}\right)$ corresponds to the intersection between the k_{esime} segment and one of the $\gamma_i(\theta, \varphi) = 0$. It is assumed that some frontier's infinitesimal deformations can be done to guarantee the existence of two and only two cutting points per segment and it is supposed that these deformations don't interfere in subsequent deductions. For a clean writing the adjacent $(l + 1)_{esime}$ and/or $(l - 1)_{esime}$ lines are named the contrasts of the $(l)_{esime}$ line.
In Fig 3 an example of segmented line and contrast is presented.

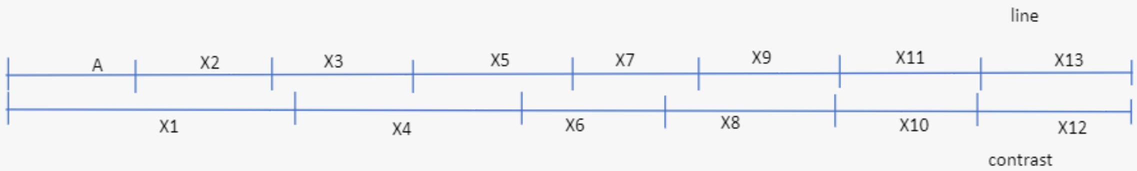

Fig 3: Line l and its respective contrast, both of them aleatory segmented.

In this model, consecutive $labels$ (segments) in the same line and in the same contrast are necessarily different from each other by definition and adjacent $labels$ between lines and contrasts may be equal or different. Equalities between $labels$ of lines and $labels$ of contrasts represent surface continuity and are related to the area of a Region. (in this model, only multiple order of equalities between $labels$ of line and $labels$ of contrast make sense, otherwise areas are not represented).
A general System of equalities and inequalities can be deduced From Fig 3 as:

$X_1 \neq A,\ or\ X_1 = A; X_2 \neq A\ and\ X_1, or\ X_2 = X_1; X_3 \neq X_2\ and\ X_1, or\ X_3 = X_1; X_4 \neq X_3\ and\ X_1, or$
$X_4 = X_3; X_5 \neq X_4\ and\ X_3,\ or\ X_5 = X_4; X_6 \neq X_5\ and\ X_4, or\ X_6 = X_5; X_7 \neq X_6\ and\ X_5, or\ X_7 = X_6$
$X_8 \neq X_7\ and\ X_6, or\ X_8 = X_7; X_9 \neq X_8\ and\ X_7, or\ X_9 = X_8;$
$X_{10} \neq X_9\ and\ X_8,\ or\ X_{10} = X_9;$
$X_{11} \neq X_{10}\ and\ X_9, or\ X_{11} = X_{10}; X_{12} \neq X_{10} and\ X_{11},\ or\ X_{12} = X_{11};$
$X_{13} \neq X_{12}\ and\ X_{11}, or\ X_{13} = X_{12}.$

Eliminating the equalities the general System can be rewritten in a most restrictive (rigid) condition (that require more $colors$) as:

$X_1 \neq A; X_2 \neq A\ and\ X_1; X_3 \neq X_2\ and\ X_1; X_4 \neq X_3\ and\ X_1; X_5 \neq X_4\ and\ X_3; X_6 \neq X_5\ and\ X_4;$
$X_7 \neq X_6\ and\ X_5; X_8 \neq X_7\ and\ X_6; X_9 \neq X_8\ and\ X_7; X_{10} \neq X_9\ and\ X_8;$

$X_{11} \neq X_{10}\, and\, X_9; X_{12} \neq X_{10} and\, X_{11}; X_{13} \neq X_{12}\, and\, X_{11}$.

A characteristic of this restricted model is that the unknowns of this system and their respective inequalities can always be written in function of the precedent unknowns along the φ axis (in the restricted φ axis when $\varphi_{final_{k-1}}$ is lower than $\varphi_{finalAdjacent_j}$ then $\varphi_{final_{k-1}}$ precedes $\varphi_{finalAdjacent_j}$. In case of φ_{final_k} is equal to $\varphi_{finalAdjacent_j}$ an infinitesimal deformation can be done to make $\varphi_{Adjacent_j}$ the succedaneous segment. A natural consequence of the precedence ordination is that 2 independently segmented lines juxtaposed are always “solvable” in 3 $colors$ because a system with only inequalities represents the worst case and can always be written as $X_k \neq X_{k-1}\, and\, X_{adjacent}$or in form of a recurrency model as $X_k(X_{k-1}, X_{adjacent})$.

In Fig 6, as $X_0 = A;\ X_1(adjacent) \neq A \Rightarrow X_1(adjacent) = B;$ $X_2 \neq A\, and\, X_1(adjacent) \Rightarrow X_2 = C$, then X_3, X_4,.. can be successively defined and solved in function of the previously calculated 2 unknowns, one coming from the antecedent segment at the same line and other from one segment at the adjacent line. Then with 3 $colors$ is always possible to paint 2 independently segmented lines juxtaposed. But, what about one hyper line along all $(0,\ 2n\pi)$ axis? Is it possible to paint it with 3 $colors$?

To represent all crossings between the Regions and the n-straight line, one hyper line and one hyper contrast are constructed and juxtaposed. The question now is to answer if 3 $colors$ are necessary and sufficient to paint a hyper line with its hyper contrast juxtaposed.

To demonstrate this, an example of hyper line and hyper contrast composed of 8 and 7 lines each is given in Fig 4:

Fig 4: Example of juxtaposition of one hyper line composed of 8 lines and its hyper contrast composed of 7 lines. This represents a 8 times replication of (θ, φ) plane in extended φ as in Fig 2.

In this model, a MAP is represented by a 2-tuple $\left(color(label_k), label_k\right)$. To utilize the previous notation $color(label_k)$ is named X_k. The deployment of the example of Fig 4 into a rigid system of inequalities can be done using the following algorithm:

Step 1: Create a n-hyper line and a n-hyper contrast and transfer the internal structure (the segments described by the 2-tuple) of each line to the hyper line and of each contrast to the hyper contrast (the structure is obtained from the intersections between the $\gamma_i(\theta, \varphi) = 0$ and the l lines of the by the n-straight line $\theta - \left(\frac{1}{(2n)}\right)\varphi = 0$ as in Fig 5.

Fig 5: Transposition of an hypothetical structure of all $\gamma_i(\theta, \varphi) = 0$ cutting points to the hyper line and to the hyper contrast of the Example.

Step 2: Enumerate unknowns of this system generating them recurrently (always generate new unknowns based on previously equationated unknowns) and transfer the unknowns to their respective positions in lines and contrasts as in Fig 6.

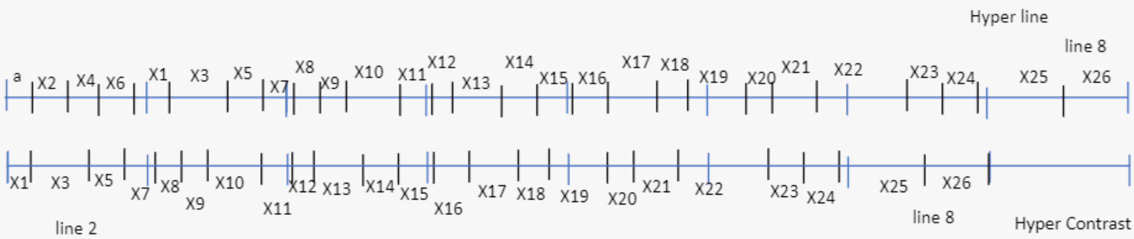

Fig 6: Deployment of Fig 5 into the structure of segments and its unknowns enumerated as a recurrent set based in the precedence of φ. The little deformation principle of frontiers is not used to evaluate its necessity!

A partial solution until four $colors$ are necessary is presented in fig 6 :

$X_1 \neq A \Rightarrow X_1 = B$; $X_2 \neq A \Rightarrow X_2 = B$; $X_3 \neq X_1 \, and \, X_2 \Rightarrow X_3 = A$
$X_4 \neq X_2 \, and \, X_3 \Rightarrow X_4 = C$; $X_5 \neq X_3 \, and \, X_4 \Rightarrow X_5 = B$;
$X_6 \neq X_4 \, and \, X_5 \, and \, X_1 \Rightarrow X_6 = A$; $X_7 \neq X_5 \, and \, X_6 \, and \, X_1 \Rightarrow X_7 = C$;
$X_8 \neq X_7 \, and \, X_3 \, and \, X_1 \Rightarrow X_7 = D$

Analyzing the result of STEP 2 in Fig 6 it is shown that new terms broken the enumeration at Hyper Line as for example X_1, X_3, X_5...etc. These terms imposes low order restrictions X_{lor} in the calculus of its antecedent term X_k which implies that the general term X_k need to be written as $X_k(X_{k-1}, X_{adjacent}, X_{lor})$. The lor index is lower than $k - 1$ index and of the Adjacent index at Hyper Line. The X_{lor} term may also appear at different positions in hyper line and in hyper contrast due to propagation of equalities between adjacent $labels$ in lines and contrasts (given that equalities between consecutive segments at the same line and at the same contrast are forbidden by the model). In true, except for the calculus of the first $color(label_k)$ all others $labels$ are X_{lor}, then $X_k(X_{k-1},\ X_{adjacent},\ X_{lor})$ wil occur naturally through the system . Obs: X_{lor} is not necessarily at $(k\ +\ 1)$ position.

Then, Step 2 must be rewritten as:

Step 2': Write the unknowns in a recurrent way. Overwrite the unknowns representing existing equalities between terms as did in Fig 6.

Step 3: Solve the unknowns recurrently $X_1; X_2\ after\ X_1; X_3\ after\ X_2$... using the letter A rather than the letter B and this rather than the letter C and this rather than the letter D. Solve the unknowns independently as until the occurrence of a X_{lor} in the calculus. When X_{lor} is involved in the evaluation of an unknown $X_k(X_{k-1},\ X_{adjacent},\ X_{lor})$, evaluate X_k and after this, check all the previous calculated values in a sense to eliminate false solutions (an example is given below).

Step 4: Return to Step 2' until all unknowns are calculated.

As $X_k(X_{k-1},\ X_{adjacent},\ X_{lor})$ represents any element of hyper line where $X_{k-1}\ and\ X_{adjacent}\ and\ X_{lor}$ are always known and in the worst case the calculated unknown $X_k is\ different\ of\ (X_{k-1}\ and\ X_{adjacent}\ and\ X_{lor})$ then this implies that 4 $colors$ are necessary and sufficient to solve all unknowns of the hyper line.

On the other side, when one unknow is solved, all the other segments having equal $labels$ assume the $color(label_k)$ of the solved unknown. In other words, when the $color(label_k)$ of one 2-tuple is calculated the $color(label_k)$ of correspondent $label_k$ is logically transmitted to all other 2-tuples that contains the $label_k$. Except for the calculated $color(label_k)$ of the first $label_k$ all other $label_k$segments receives the same $color(label_k)$ and all these segments are transformed into long order restriction X_{lor}along the hyper line, creating the condition that forces the system to have 4 $colors$ to be solved. This procedure is repeated at each new unknown calculated.

To a better understanding of the behavior of a system composed of equalities and inequalities, a new Example composed of two lines and contrasts is proposed in Fig 7.

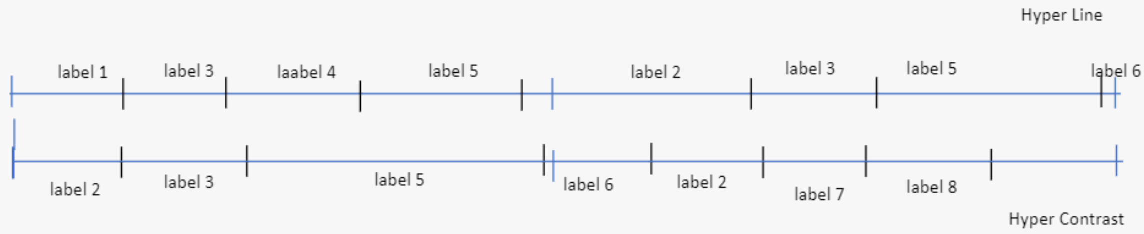

Fig 7: Example of a general hyper line and hyper contrast brought from one planar MAP. Observe the equalitie's way through the lines and contrasts of the model.
Now a general System can be written as in Fig 8, using X_k as unknowns. The precedence rule is applied to order the unknowns and equalities between hyper line and hyper contrast are represented too. The 2-tuple $\left(color(label_k), label_k\right)$ is represented by X_k.

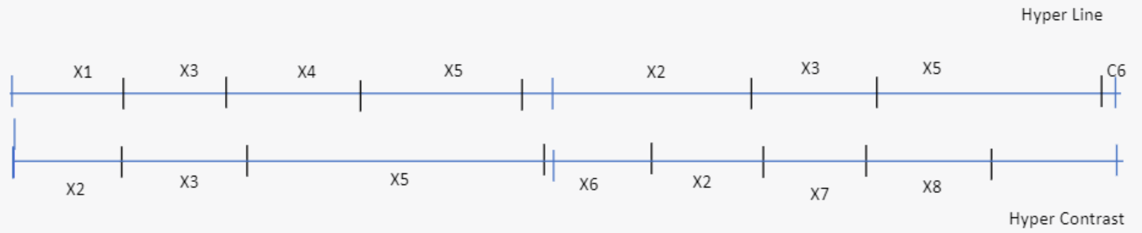

Fig 8: As in the previous examples the $colors$ and the $labels$ are transcripted $\left(X_k\right)$ to the structure of the hyper line and hyper contrast following the precedence rules.

Applying the precedence Rules to the system of Fig. 8 there is:

$X_1 = A; X_2 \neq X_1 \Rightarrow X_2 = B; X_3 \neq X_2 and\ X_1 \Rightarrow X_3 = C; X_4 \neq X_3 \Rightarrow X_4 = A\ or\ B*;$
$X_5 \neq X_2 and\ X_3\ and\ X_4 \quad \Rightarrow X_5 = A*; X_6 \neq X_5\ and\ X_2 \Rightarrow X_6 = C;$
$X_7 \neq X_2\ and\ X_3 \Rightarrow X_7 = A; X_8 \neq X_7\ and\ X_5\ and\ X_3 \Rightarrow X_8 = B$

1-) Observe that, once one $label$ is determined, all segments with same $label$ receive the same $collor$ and except for the first one they impose long order restrictions along the hyper line.
*2-) The calculated initial values of X_4 were $A\ or\ B$. When the calculus was propagated until the first X_{lor} the value $X_4 = A$ implied in the inclusion of a fourth $color\ D$ to solve the problem while the use of $X_4 = B$ reduce the partial solution to tree $colors$. Because of this, $X_4 = A$ was eliminated from the possible solution's Set.

As considered and written before, each segment is a 2-tuple $(color(label),\ label)$.

Then, once the problem is solved in $color$ Space, attribute $colors$ to $labels$ is immediate. In the case above $color(label_1) = A; color(label_2) = B; color(label_3) = C;$ $color(label_4) = B; color(label_5) = B; color(label_6) = C;$ $color(label_7) = A;\ color(label_8) = B.$

Then, the problem of painting a MAP is transformed into a resolution of a System containing equalities and inequalities and this System always has a solution. As a recurrence form the solution of the problem normally evolves as a tree and as the X_{lor} are presented the previous signature must be reviewed and some previously assigned values must be discharged as was seen in *2-).

A final consideration

The power of this model is based on precedence rules as presented in Fig 9

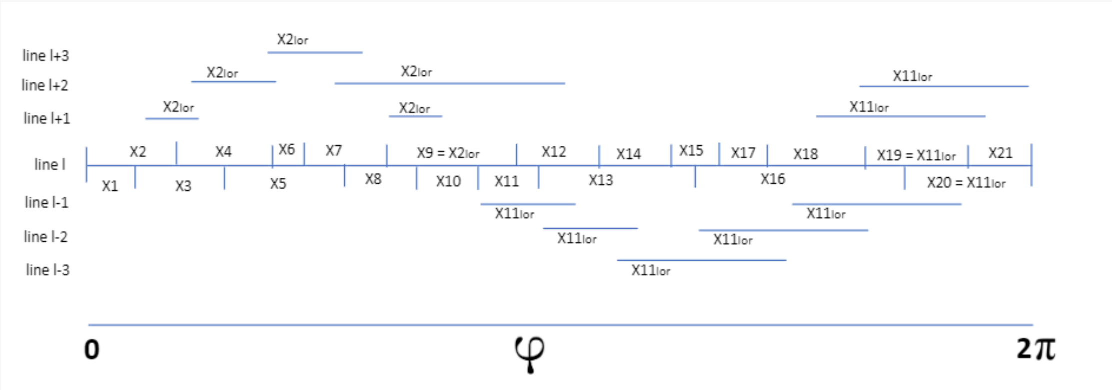

Figure 9: Resume of concepts. Only segments corresponding to equalities X_{2lor} and X_{11lor} are represented at lines $(l + / - k)$.

All deductions in the article are related to this simple figure (concepts) where the X_i are enumerated according to their position related to φ angle. The propagation of equalities begins in X_2 and X_{11} and evolves between lines as long order restrictions terms that are represented by X_{2lor} and X_{11lor}. With these concepts established all deductions given throughout the text are immediate.

Hint: The importance of the continuity in MAPs can be viewed when Rule B and Rule D are violated as in these two **non MAPs** that are presented in Fig 10

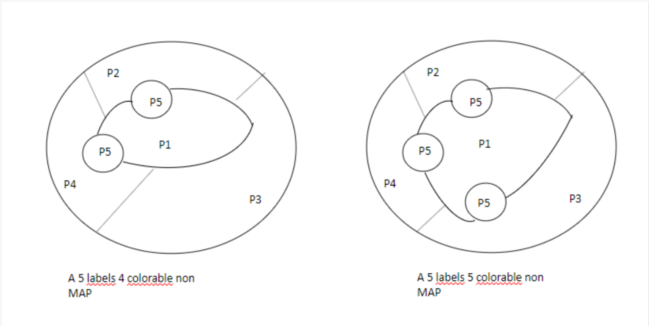

Fig 10: Two non MAP examples

Figure 10 illustrates instances of Rule B and D violation, manifesting in both (5 $labels$) 4 $colorable$ and (5 $labels$) 5 $colorable$ non-MAPs. Notably, these non-MAPs can be resolved with four and five $colors$ respectively. Examining Figure 10 yields the inference that, beyond serving as a viable coloration scheme for MAPs, four $colors$ also prove effective for certain categories of non-MAPs.

Moreover, it is evident that in function of the complexity, non-MAPs may necessitate five or more $colors$ to be painted. Instances abound where 6, 7, or even more $colors$ are necessary for the proper coloration of non-MAPs.
The exploration of non-MAPs may be intriguing subjects for study.
As an aside, out of curiosity, several world maps were consulted on the internet to examine how the $colors$ of the USA and Alaska are usually represented. No consensus was discerned, with both identical and diverse colorings encountered. While this observation may give rise to curious questions, it falls outside the purview of the current investigation.

And why is any planar MAP four colorable?
Five colorable X_l terms needs to be redefined as $X_l(X_{l-1}, X_{adjacent}, X_{lor1}, X^*_{lor2})$. In this way the general term X_l may be compared with $X_{l-1}, X_{adjacent}, X_{lor1}\, and\, X^*_{lor2}$ and can receive the fifth $color$ in case all of them are different.

In fig 11 is presented a basic graphic representation of the general term $X_l(X_{l-1}, X_{adjacent}, X_{lor1}, X^*_{lor2})$.

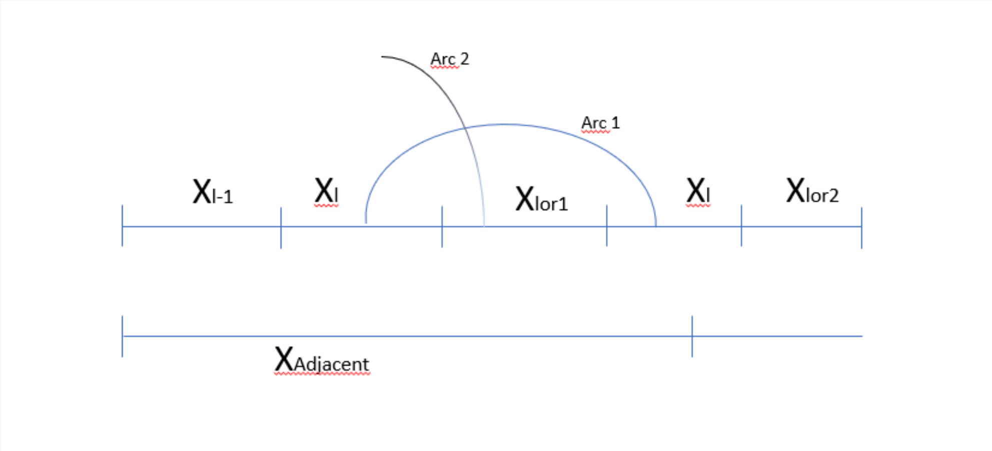

Fig 11: Physical representation of a 5 $color\ X_l$ term.

From Fig 11 it can be seen that,:

a-) X_{lor1} is disconnected of X_{l-1} and X_{lor2} because that, **imposed by the continuity of Regions**, there exists an "arc1" connecting the two $X_l\ labels$. The arc1 is a continuous path representing the equalities propagation between the $X_l\ labels$ corresponding to successive contrasts. As X_{lor1} is separated by arc1 from X_{l-1} and of X_{lor2}, them $color(X_{lor1})$ may be equal to $color(X_{l-1})$ or $color(X_{lor2})$ and the solution reduces the number of $colors$ to four.

OR

b-) X_{lor1} is connected through arc2 to an external region then, analyzing the Fig 11 it can be observed that:

b1-) it can be seen that if arc 2 is continuous, there is an intersection between arc1 and arc2 which implies that there are 2 different labels associated with the same Region and this violates Rule C then this is false by definition or,

b2-) if arc 2 is discontinuous this violates Rule B that establishes that Regions are, by definition, always internally connected then the existence of a discontinuous arc 2 is false.

Then, the existence of an arc2 is denegated.

Then, planar MAPs are always paintable with Four $colors$.

Conclusion

The proposal of this article was to present a simple demonstration of the FOUR COLOR THEOREM. The segments ordinance, the concept of precedence in φ, the evaluation of $X_{calculated}$ as a deterministic function of $(X_{k-1}, X_{adjacent}, X_{lor})$ and the recurrent solution of $X_{calculated}$ were good surprises not previewed when the model was created. In b2-), based on the analysis of X_l term, the implication of the topological continuity with the solution to the problem is established for the first time. There are other discovers related to continuity in θ that are not reported because it doesn't seem important to the problem.

As initially desired it was possible to demonstrate that the Four Color Theorem is true using simple mathematical tools, basically covering a MAP with a straight line. In benefit of a pleasant reading not all arguments are extensively explained. In case of doubts, a reading of Jansen 2021 and 2023 article's can give a better understanding of the present article.

The author's hope is that the presented ideas contribute to an additional demonstration of the Four Color Theorem and can inspire researchers to contribute with other types of formal solutions. It is a expectative of the author that mathematicians can do a much better work based on these and previous ideas. Certainly there are many other ways to demonstrate this still fascinating problem.

The author recognizes his limitation with pure mathematical formalism and is the only responsible for the existing errors and mistakes contained in the article. Thanks to his relative ignorance with mathematical formalism the author feels relatively free to commit some atrocities and expects to have not written a nonsense article.

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