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https://doi.org/10.1590/SciELOPreprints.7216

Submitted on: 2023-10-31
Posted on: 2023-11-07 (version 1)
(YYYY-MM-DD)

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Abstract

The Four-Color Theorem originated from the attempt to solve the problem of painting MAPS over a plane or spherical surface. Over a century and a half, this problem underwent various abstractions until it was resolved in 1976. The proposed solution, which is disruptive, computationally calculates the number of possible states for a representation of a flat map. Although it is resolved, the lack of a formal proof for this problem causes some discomfort. Therefore, a solution that uses more traditional techniques and is easily understandable is needed. In a previous article, a solution based on equalities and inequalities between boundaries was presented. Now in this article, a generic spheroidal MAP is subjected to various one-to-one relationships until a generator of all possible MAPS on a two-dimensional surface partitioned into n^2 cells are found. Four-Colors are proved to be necessary and sufficient to paint a two dimensional MAP. It is explained at the end of the article that the imposition of a fifth color as a necessary condition implies a contradiction.

Keywords: Four-Color Theorem; discrete method solution; Discriminant function
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Introduction

The problem of the four colors (1) arose in 1852 when Francis Guthrie observed that four colors seemed to be sufficient to color any map of England. After a long process his conjecture was sent to Arthur Cayley, who published the problem in the Philosophical Magazine in 1879, and the problem gained international notoriety.

In 1878, Alfred Kempe published an alleged proof that was accepted until Percy Heawood found a subtle flaw in Kempe’s proof. On the other hand, Heawood is able to prove that five colors are always enough.Heawood then proved the Five-Color Theorem, which asserts that any flat map can be colored with five colors.

From 1890 until 1976 several mathematicians attempted to find a new proof or a counterexample and large areas of graph theory (perfect graphs, chromatic polynomials, . . .) are developed in an attempt to prove the 4CTfor the Four-Color Theorem but all of them failed

In 1977 Kenneth Appel and Wolfgang Haken published the first widely accepted proof of the 4CT generating controversy due to the use of a computer in the demonstration.

In 1997, Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas presented a new proof of the Four-Color Theorem simplifying Haken's proof. They also used a computer program to do the demonstration.

In 2008, Georges Gonthier and Benjamin Werner published a proof using a computer-assisted proof system called Coq, formalizing Robertson et al.'s proof.

In 2012, Thomas Hales announced a project called Flyspeck, which aimed to formalize the proof of the Four-Color Theorem using another computer-assisted proof system called HOL Light. The project was completed in 2014 and published in the Forum of Mathematics Pi in 2017.

In 2016, Marijn Heule, Oliver Kullmann, and Victor Marek presented a proof of the Four-Color Theorem using a resolution-based problem-solving method based on Boolean satisfiability (SAT). They reduced the problem to 633 Boolean formulas, each representing an unavoidable configuration, and used a SAT solver to verify their colorability. The method took only a few minutes to execute and filled about 200 printed pages. The proof was published in the Journal of Automated Reasoning in 2017.

The purpose of this work is to return to the origin of the Four Color Problem and use old mathematical tools contained in good Calculus books (Arfken, Courant) to solve it..
To paint a MAP on a surface means delineating countries (Regions, oceans, etc.) and painting them so that neighboring countries (neighboring Regions) have different $colors$. Due to the restriction of belonging to a surface, these are usually named planar MAPS and from now on in this text they will be named MAPS. In a MAP, each country is characterized by three variables, namely: the area of the country itself (which is abstracted from the problem); a $label$ (name of a Country, Region, ocean, etc.), and a $color$. In the statement of T4C, there is an explicit and Necessary condition that is:
C-) Neighboring Regions have different $colors$;
and there are other implicit hypotheses which are:
A-) There is territorial continuity within each Region (which is related to the continuity in R^2 inside a closed surface bounded by a contour). In other words, within the same Region (Country), there is always a continuous path that connects two different points within the same Region (Country);
B-) Each Region has a single $color$ and a single $label$;
C'-) Neighboring Regions have different $labels$;
D-) Non-neighboring Regions can have the same $color$, but they must necessarily have different $labels$.
The Four-Colors Theorem consists in demonstrating that it is possible to paint any MAP with 4 $colors$.
It will be demonstrated that 4 $colors$ are necessary and sufficient to paint a MAP in 10 steps.

Step 1 - Spherical Surface Discretization

Consider a MAP on a spherical surface in the coordinates $(r = 1, \theta, \varphi)$. The spherical surface can be partitioned into curved cells of size $(\Delta\theta, \Delta\varphi)$ where each cell belongs to only one Region, except for the ones that contain a boundary segment*. In this step, it will be used without proof the fact that the area bounded by a closed curve on any 2-variable surface can be approximated with any level of precision by a discrete summation (in theory,

$\int\int f(x,y)dxdy \approx \sum\sum f(\xi_i,\eta_j)\Delta x_i \Delta y_j$ when $\Delta x_i \rightarrow 0$ e $\Delta y_j \rightarrow 0$)). By the same argument, it is assumed that the area occupied by cells containing boundaries also tends to 0. Therefore, a Region is represented by the union of cells with the same characteristics $(color, label)$, and a MAP is represented by the union of all the Regions that constitute it. This discrete representation approximates the continuous representation with any desired level of error.

Step 2 - First one-to-one relationship.

There is a one-to-one relationship between the spherical surface and the coordinate plane (θ,φ), where $0 \leq \theta < \pi$ and $0 \leq \varphi < 2\pi$. The closed curves on the spherical surface are represented in the plane (θ,φ) by equations $\psi_i(\theta,\varphi) = 0$. From this point onward, the plane of variables (θ,φ) will be used to represent any MAP.

The (θ,φ) plane can be discretized into rectangular cells where $\Delta\theta = \frac{\pi}{n}$ and $\Delta\varphi = 2\frac{\pi}{n}$, where n is adopted as an even number that tends to infinity. As previously presented, it is assumed without proof** that the area bounded by a closed curve $\left(\psi_i(\theta,\varphi) = 0\right)$ can be approximated with any level of precision by the union of their inside cells $(\Delta\theta, \Delta\varphi)$ provided that $n \rightarrow \infty$ and consequently $\Delta\theta \rightarrow 0$ and $\Delta\varphi \rightarrow 0$. Similarly to a matrix, the cells in the plane (θ,φ) can be enumerated by associating a row index i to the variable θ and a column index j to the variable φ. The cells can then be indexed by the indices i and j and be represented by $C_{i,j}$. Each cell $C_{i,j}$ is associated with the ordered pair $(color(C_{i,j}), label(C_{i,j}))$.

Step 3 - Second one-to-one relationship

A one-to-one relationship can be established between the cells $C_{i,j}$ of a matrix M and the elements v_k of a vector Vector V_k, simply by making $v_k = C_{i,j}$ for all $k = n * i + j$ (in the specific case of $colors$, $color(v_k) = color(C_{i,j})$. Disconsidering null measurement elements , two Maps are different when they differ by a finite area A. In this case, one can always adopt a sufficiently large number n that guarantees that at least one cell is entirely contained in A. In the same way, as n approaches infinity any MAP is characterized by a single Vector V_k.

Step 4 - A simple way to demonstrate that Four Colors are a necessary condition.

One way to demonstrate that 4 $colors$ are necessary to paint a MAP is to provide an example. Consider the example given in Figure 1:

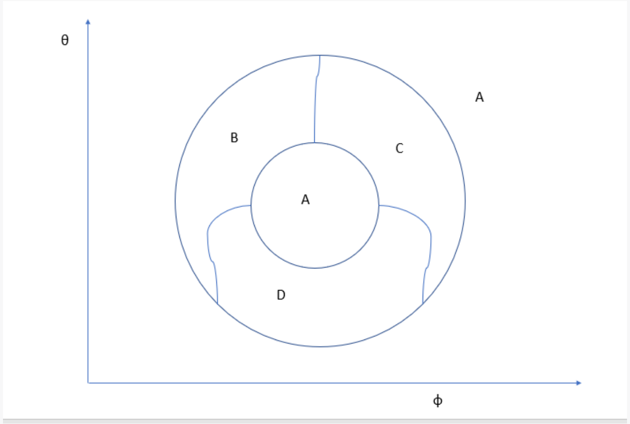

Figure 1: Example of a MAP with 4 $colors$

According to the .conditions A-), B-), C-), C'-) and D-) it turns evident that 4 $colors$ are necessary to paint the MAP of Figure 1.

Step 5 - Topological Structure of a Discretized Path

In the model to be adopted, the $color(C_{i,j})$ of each cell is related to the $colors$ of its neighbors $color(C_{i-1,j})$, $color(C_{i,j-1})$, $color(C_{i+1,j})$ and $color(C_{i,j+1})$, (therefore, in this model, $C_{i-1,j-1}$, $C_{i+1,j-1}$, $C_{i+1,i+1}$ and $C_{i-1,j+1}$ are not neighbors of $C_{i,j}$). To extend the concept of connectivity to the discrete systems given by the matrix $C_{i,j}$ (which approximate continuous MAPS described on a spherical surface with any degree of precision), it is necessary to create the concept of a distance between cells. Let $d_{i,j}^{l,m}$ be the "distance" between two cells $C_{l,m}$ and $C_{i,j}$ given by $d_{i,j}^{l,m} = \left||l - i| + |m - j|\right|$. As can be seen, neighboring cells have $d_{i,j}^{l,m} = 1$.

Painting a connected Region (a Country), signifies that for a given cell $C_{i,j}$, at least one of the cells $C_{i+1,j}$ or $C_{i-1,j}$, or $C_{i,j+1}$ or $C_{i,j-1}$has the same $color$ and $label$ as $C_{i,j}$ (meaning that they are neighboring cells ($d_{i,j}^{l,m} = 1$) with the same $color$ and with the same $label$).

If two neighboring cells ($d_{i,j}^{l,m} = 1$) have different $colors$, this indicates that they do not belong to the same Region, and, by definition, these cells have different $labels$. In the context used here, these Regions are disconnected since there is no connected path that unites 2 points into different Regions.

In other words: a-) If a group of cells has $d_{i,j}^{l,m} = 1$ and have the same $label$, they are neighbors and they are in the same Region (the same Country, that is a connected set) and, therefore, they have the same $color$; b-) If a group of cells have $d_{i,j}^{l,m} = 1$ and different $labels$, they are neighbors but they are in different Regions and, therefore, they have different $colors$. Notice that a-) and b-) remain true if the words $labels$ and $colors$ are interchanged. c-) If two cells have $d_{i,j}^{l,m} > 1$, they are not neighbors then they are in different Regions. These cells necessarily have different $labels$ but they can have the same $color$ since they are not neighbors.

As the indexes $l = n * k_1 + m$ and $i = n * k_2 + j$ may be attributed to v_l and v_i, it can also be defined a distance d_i^l as $d_j^l = \left|\left|k_1 - k_2\right| + |m - j|\right|$ (distance between two elements v_l e v_i) for $k_1 = quotient(l/n)$; $k_2 = quotient(i/n)$; $m = remainder(l/n)$ and $j = remainder\ (i/n)$.

Extending the concept of distance between cells $C_{i,j}$ to the elements of the Vector V_k, the $labels$ of v_k agree to three rules derived from the topological continuity of the (ϑ, φ) space:

a-) If $d_k^{k+s} = 1$ and $label\left(v_k\right) = label\left(v_{k+s}\right) \Rightarrow v_k$ and v_{k+s} are neighbors and they are in the same Region, $\Rightarrow Color(v_k) = Color(v_{k+s})$;

b-) If $d_k^{k+s} = 1$ and $label\left(v_k\right) \neq label\left(v_{k+s}\right) \Rightarrow v_k$ and v_{k+s} are neighbors but they are not in the same Region, $\Rightarrow Color(v_k) \neq Color(v_{k+s})$. Notice that a-) and b-) remain true if the words $labels$ and $colors$ are interchanged and

c-) If $d_k^{k+s} > 1 \Rightarrow v_k$ and v_{k+s} are not neighbors and then they are not in the same Region, $\Rightarrow label\left(v_k\right) \neq label\left(v_{k+s}\right)$ and $Color(v_{k+s})$ is not necessarily different from $Color(v_k)$.

Step 6 - Painting a MAP

If the axes (θ, φ) are each divided into n parts, then there are n^2 cells (rectangular sections) covering the entire (θ, φ) plane . There can be a maximum of n^2 different $labels$ (Countries) in this MAP, and there can be up to n^2 different $colors$ (all cells with different $colors$).

Let the $color$ of each cell $C_{i,j}$ be given by the ordered pair $color(C_{i,j} = \left(-1^i, -1^j\right))$. For this each cell has a different $color$ from its adjacent cells. The Map has n^2 distinct Regions and contains n^2 different $labels$ (Countries), and this MAP is painted with the 4 $colors$ $\{(-1, -1), (-1, 1), (1, -1)$ and $(1, 1)\}$. The same $colors$ $\{(-1, -1), (-1, 1), (1, -1)$ and $(1, 1)\}$ can be applied to the Vector V_k by associating the $color$ of element v_k with the $color$ of cell $C_{i,j}$ in the form $color(v_k) = color(C_{i,j})$ according to the index relation $k = n * i + j$.

Step 7 - Third one-to-one relationship

The ordered pairs $\{(-1, -1), (-1, 1), (1, -1)$ and $(1, 1)\}$ can be replaced by the numerals {1, 2, 3, and 4}, which are the digits that make up a base-4 numeral.

Step 8 - Fourth one-to-one relationship

Assigning the numbers {1, 2, 3, and 4} to the elements v_k of Vector V_k, it is observed that the juxtaposition of these elements of the vector represents a base-4 numeral with n^2 digits. For each matrix Mi,j with n^2 independent elements, there corresponds a different painting of a MAP with n^2 cells. All possible $color$ representations of V_k are given by a n^2 digit base-4 numerals and they are given by the 4^{n^2} numbers contained between $\{1..1\}$ and $\{4...4\}$ This set is complete (there are no gaps) and is the space of all possible mappings of 4 $colors$ onto n^2 cells.

Step 9 - Proof that 4 Colors are sufficient to paint a MAP

Given the $colors$ $\{1, 2, 3, 4\}$, one and only one of these $colors$, can be assigned to the element $color(v_k)$. So $color(v_k) = \{1 \ \ or\ 2 \ \ or\ 3 \ \ or\ 4\}$. Let any base-4 numeral with n^2 digits be given. This numeral contains sequences $\{(1..1)(2..2)(3..3)(4..4)\}$ Each subsequence has $d_i^l=1$, so each sequence has one $label$.

For a general n^2digit 4-base number it follows that:
d-) If two Sequences l and i have the same $color$ and there are elements v_l and v_i such that $d_i^l = 1 \Rightarrow v_l$ and v_i belong to the same Region and therefore they have the same $label$ (observe the compliance with Items A and B);
e-) If two Sequences l and i have different $colors$, $color(v_l) \neq color(v_i)$ and $d_i^l = 1 \Rightarrow v_l$ and v_i are neighbors and they belong to different Regions, so ⇒ they have different $labels$ (observe the compliance with Item C), and finally
f-) If two Sequences l and i have $d_i^l > 1$ they are not neighbors and, therefore, they have different $labels$, but they can have anyone of the 4 $colors$ (observe the compliance with Item D).
The set of all base-4 sequences that have as a total of n^2 digits (ranging from $\{11..1\}$ to $\{44..4\}$) is a complete set and there is no hole in it. All possible combination of $colors$ are contained between these limits. By the other side, the d_i^l metric allows to distinguish between internally connected Regions assigning them the same $label$, and to their neighbors, properly assigning their correct $labels$. Then, as to each n^2numeral sequence there corresponds one MAP with a related set of $labels$. (Observe that there is a minimum of 4! natural redundancy on $colors$ due to the set of permutations between the $Collors$ $\{1, 2, 3, 4\}$). Then, Four colors are sufficient to paint a MAP. If Four $colors$ are necessary and are sufficient, then the Four Colors Theorem is demonstrated.

Step 10 - But what about Painting a MAP with 5 $Colors$?

Finally, what happens if 5 $colors$ are used to paint a MAP? It is obvious that if 4 $colors$ are necessary and sufficient to paint one specific MAP, then 5 $colors$ are sufficient to paint the same MAP. but what happens to the relation between the labels when the use of 5 $colors$ is

a necessary condition?
To put this problem, assume by Hypothesis that 4 $colors$ are necessary and are not sufficient. Then suppose as Thesis that 5 $colors$ are necessary and sufficient to $color$ a MAP.
To see what happens in a Four $Color$ MAP suppose a n^2 sequence composed of k subsequences
$\{(11..1)_1(33..)_2(44..)_3(22..)_4(11..)_5(44..)_6(22..)_7(11..)_8(33..)_9(44..4)_{10}(11..)_{11}...(33..)_k\}$
To these subsequences it can be applied a correspondent sequence of k $labels$
$\{label_1 label_2 label_3 label_4 label_5 label_6 label_7 label_8 label_9 label_{10} label_{11}.. label_k\}$

Now, using the discrimination property of d_m^l on each pair elements v_l and v_m belonging to the subsequences $(\,)_i$ and $(\,)_j$, some equalities between $labels$ can be derived for example as $\{label_1 label_2 label_3 label_4 label_1 label_6 label_4 label_1 label_2 label_{10} label_{11}.. label_2\}$ (obs that it is not necessary analyze the distance d_m^l inside a $(\,)_i$ subsequence. All elements have $d_m^l = 1$ and belongs to $label_i$) . Suppose now the existence of a fifth $color$ 5 as a necessary condition to substitute the contents of $label_{11}$. **Then, if $(11..)_{11}$ needs to be substituted by $(55..)_{11}$ as a necessary condition this means that $(11..)_{11}$ is a neighbor of another $(11..)_r$ subsequence**, **but this contradicts the discriminant function d_m^l that says that $(11..)_{11}$ or belongs to the same $label$ as some other $(11..)_j$ subsequence or it is not a neighbor of a $(11..)_m$ $label$ leaving to a contradiction!**
This assertion was so difficult to understand as it is simple to write.
As this analysis is true for any $(..)_n$ subsequence then existence of a fifth $color$ 5 as a necessary condition leaves to a contradiction with the discriminant d_m^l classification and then **the Thesis is denegated and Four $Colors$ are necessary and sufficient to paint any MAP .**
Finally 5 (or more) $colors$ are sufficient to paint a four $color$ MAP but they are not necessary.
As Step 4 and Step 10 are almost independent of the other Steps they may constitute a two page demonstration of the Four Color Theorem (this assumption requires more work to be done).
With this, the Four-Color Theorem is demonstrated.

Conclusion:

- Given any 4-base n^2 digit number, the function d_i^l is a discriminant of this 4-base n^2 digit and it permits to separate the arrangement of $Collors$ into Regions according to their relationship. It guarantees the $labels$ unicity and the adequate identification of $labels$ and their neighboring relations. The use of the discriminant d_i^l in any given sequence of $colors$ described by a n^2 4-digit base numeral will separate them into different Regions respecting the continuity of each one and according to the $color$ restrictions imposed between neighbors !
- It has been demonstrated that, at least for the model employed, the concept of continuity (in R^2) needs to be considered at some stage of the solution of the Four

Color Theorem. Models. Eliminating the R^2 continuity concept in the solution's strategy needs care.

- It has been demonstrated that 4 $colors$ are necessary and sufficient to paint a general two dimensional MAP .
- It has been demonstrated that all n^2 cells MAPS configurations existing over a spherical surface $(r = 1, \theta, \varphi)$ are represented by the 4^{n^2} elements of 4-base n^2 sequence numerals, since each one of these n^2 sequence is independent of the others.
- It has been demonstrated that letting n→ ∞ any continuous MAP can be approximated by a discrete n^2 MAP.
- It has been demonstrated that 5 $colors$ are sufficient to paint a Four Color MAP, but if 5 $colors$ are necessary, this lead to an internal contradiction that leads to the acceptance of 4 $colors$ as necessary and sufficient conditions to paint a MAP.

The author's hope is that the presented ideas contribute to an additional vision of the Four Colors Theorem and can inspire researchers in other types of formal solutions . As a non-mathematician the author recognizes his limitation with a pure mathematical formalism. The author is the only responsible for all the existing errors and mistakes contained in the article.

Acknowledgements to the patience, corrections and suggestions done by the Chemical Engineering professor Dr. André Antunha Gonçalves and by the Inorganic Chemistry professor Dr. Juliano Alves Bonacin and to the ones who introduced me into this problem.

* As n→ ∞, the area occupied by the boundaries tends to zero. This result is well established and used as for example in the definition of entropy. The entropy is obtained by the sum of isothermal and isentropic cycles then $S_{total} = \sum S_{internal\ cicles} + \sum S_{boundary\ cicles}$. When n→ ∞, $\sum S_{internal\ cycles} \rightarrow S$ and $\sum S_{boundary\ cycles} \rightarrow 0$.

** By the definition of Area = $lim_{n\rightarrow\infty} \sum_{i,j=1}^{n} \Delta x_i \Delta y_j$.

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The author has no conflict of interest to declare.