Regularized determinant of the Laplacian on forms over odd dimensional projective spaces

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Regularized determinant of the Laplacian on forms over odd dimensional projective spaces

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Abstract

We establish formulae for the regularized determinant of the twisted Laplacians on forms over odd dimensional real projective spaces. This work corresponds to a generalization of the previous formula for this type of space and we prove the equivalence in the common cases, what leads to interesting, if simple, identities involving special values of Bernoulli polynomials and the Riemann zeta function. As application, we calculate the Analytic Torsion of these spaces in relation to all unitary representations of their fundamental group.
MSC2020: 11M36, 11B68, 11M06. **Keywords** Regularized determinants; projective spaces; Bernoulli polynomials; Riemann zeta function.

1 Introduction

Regularized determinant is a concept that arises from the study of the spectral zeta functions and has applications on areas such as analytic number theory, global analysis and physics.

The spectral zeta functions were established by Minakshisundaram and Pleijel [10] as special Dirichlet series constructed in terms of the eigenvalues of the Laplacian on functions over compact differentiable Riemannian manifolds. Since these functions revealed themselves to be useful on a wide range of applications, they were appropriately generalized to several contexts.

In this text we consider the spectral zeta function $\zeta(s, \Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\rho})$ of the twisted Laplacians $\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\rho}$ associated to unitary representations $\rho : \pi_1(\mathbb{R}\mathrm{P}^{2m-1}) \to U(N)$

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of the fundamental group of the odd dimensional real projective spaces. The twisted Laplacians are a generalization of the Laplace-Beltrami operator Δ_M^q, since Δ_M^q is essentially the twisted Laplacian associated to the trivial representation of $\pi_1(M)$. Denoting the eigenspace associated to an eigenvalue $\lambda \in \mathbb{R}$ by $E(\lambda, \Delta_{M,\rho}^q)$, the zeta function of $\Delta_{M,\rho}^q$ is defined for $\mathrm{Re}(s) > \frac{\dim(M)}{2}$ by:

$$\zeta(s, \Delta_{M,\rho}^q) = \sum_{\lambda \neq 0} \dim(E(\lambda, \Delta_{M,\rho}^q))\lambda^{-s},$$

and is extended meromorphically to $\mathbb{C}$, being analytic at $s = 0$, see [4, Theorem 1.12.2].

The importance of the spectral zeta functions $\zeta(s, \Delta_{M,\rho}^q)$ relies on their intrinsic relation to the Riemann zeta function and the several applications they have, in special providing useful informations about the underlying manifold. Some of these informations come from the study of the regularized determinant of the twisted Laplacian, $\det \Delta_{M,\rho}^q$, which is obtained from the derivative at zero of $\zeta(s, \Delta_{M,\rho}^q)$.

The regularized determinant of $\Delta_{M,\rho}^q$ is defined in terms of $\zeta(s, \Delta_{M,\rho}^q)$ by:

$$\det \Delta_{M,\rho}^q = e^{-\zeta'(0, \Delta_{M,\rho}^q)},$$

in analogy to the fact that for a diagonalizable invertible linear operator $T : V \to V$ acting on a finite dimensional vector space, the auxiliary function

$$Z(s,T) := \sum_{\lambda} \dim(E(\lambda, T))\lambda^{-s}$$

satisfies:

$$\det T = e^{-Z'(0,T)}.$$

One of the most important ways the regularized determinant provides informations about the underlying manifold is through the study of the Analytic Torsion $T_a(M,\rho)$, which is defined for the representation $\rho : \pi_1(M) \to U(N)$ by the expression:

$$T_a(M,\rho) = \sqrt{\prod_{q=0}^{n} \left(\det \Delta_{M,\rho}^q\right)^{q(-1)^{q+1}}} = \exp\left(\frac{1}{2}\sum_{q=0}^{n}(-1)^q q \zeta'(0, \Delta_{M,\rho}^q)\right).$$

For representations $\rho : \pi_1(M) \to U(N)$ for which the associated elliptic complex is acyclic, i.e., has trivial cohomology, the Analytic Torsion is a topological invariant of the manifold and the representation which can detect non-homeomorphic manifolds even when their fundamental group, homology and cohomology are the same.

The reason for studying the Analytic Torsion of the twisted Laplacian is the fact that the elliptic complex associated to the trivial representation of the fundamental group never has trivial cohomology, which means the Analytic Torsion associated to the regularized determinant of the Laplace-Beltrami operator is not an invariant of the manifold in general. In particular for odd dimensional real projective spaces the elliptic complex associated to a unitary representation $\rho : \pi_1(M) \to U(N)$ is acyclic if, and only if, the representation may be decomposed in the form $\rho \simeq \sigma_2 \oplus \cdots \oplus \sigma_2$, where σ_2 denotes the non-trivial irreducible representation of $\pi_1(\mathbb{R}\mathrm{P}^{2m-1})$.

In this text we establish explicit formulae for the regularized determinant of the twisted Laplacian on odd dimensional real projective spaces and establish some related results.

On section 2 we use the decomposition of $C^\infty\left(\Lambda S^{2m-1}\right)$ into $SO(2m)$-irreducible modules, as well as the association of eigenvalues to each of these modules established on [7] and the Theory of [5, section 3.9] to decompose $C^\infty\left(\Lambda\mathbb{R}\mathrm{P}^{2m-1}\right)$ into $SO(2m)$-irreducible modules and since the dimension of each $SO(2m)$-irreducible module was established previously on [15], we obtain the spectrum and dimension of the eigenspaces of $\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}$, and consequently, we get this information for $\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\rho}$ for any unitary representation ρ of the fundamental group of $\pi_1(\mathbb{R}\mathrm{P}^{2m-1})$.

On section 3 we establish our formulae for determinants which corresponds to a generalization of the formula obtained by Hartmann and Spreafico on [6] and follows the same method adopted by Weng and You on [15] and by the author on [12] and [13].

Section 4 is dedicated to the proof of the equivalence between our formula and the one established by Hartmann and Spreafico on [6] in the cases covered by the second one, what leads us to interesting identities involving special values of Bernoulli polynomials and special values of the Riemann zeta function.

On section 5, as an application, we use our formula to calculate the analytic torsion of $\mathbb{R}\mathrm{P}^{2m-1}$ in relation to all unitary representations of $\pi_1(\mathbb{R}\mathrm{P}^{2m-1})$, what was done before, for example on [8], but using different approaches.

Moreover, for the convenience of the reader, we have added an appendix where we enlist results from other references that are fundamental for proving some of the results present in this manuscript.

2 Spectrum and eigenspaces of the twisted Laplacian

To establish the formulae for determinants on odd dimensional real projective spaces we first describe the spectrum and dimension of the eingenspaces of the Laplacian on forms over $\mathbb{R}\mathrm{P}^{2m-1}$ and them we use these informations to study the associated spectral zeta functions. Since S^n is the universal covering space of $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$, we use the already known data for the Laplacian on $C^\infty(\Lambda^* S^n)$ to deduce the desired results on $C^\infty(\Lambda^*\mathbb{R}\mathrm{P}^{2m-1})$, $m \geq 2$; and for the case of $C^\infty(\Lambda^*\mathbb{R}\mathrm{P}^1)$, we use Hodge duality and the already known data for the Laplacian on $C^\infty(\Lambda^0\mathbb{R}\mathrm{P}^1)$. We outline now the main results that we need to accomplish our first objective for $n \geq 2$.

On [7], Ikeda and Taniguchi decompose the spaces $C^\infty(\Lambda^*(S^n))$ of forms over S^n into irreducible $SO(n+1)$-modules and describe the eigenvalue associated to each of them, which unravels the spectral decomposition of these spaces. On section 6 of their text, they describe the relationship between eigenforms on S^n and harmonic polynomial forms on $\mathbb{R}^{n+1}$.

Let $\bar{\Delta}$ denote the Laplace operator on $C^\infty(\Lambda^*\mathbb{R}^{n+1})$ and let δ_0 be the formal adjoint of the exterior differential on this space. For $q \in \{0\} \cup \mathbb{N}_n = \{0\} \cup \{i \in \mathbb{N} \mid i \leq n\}$ and $k \in \{0\} \cup \mathbb{N}$, the space H^q_k of harmonic polynomial forms on $\mathbb{R}^{n+1}$ is the space of $\alpha \in C^\infty(\Lambda^q(\mathbb{R}^{n+1}))$ that satisfy:

$$\alpha = \sum_{0 \leqslant i_1 < \cdots < i_q \leqslant n+1} \alpha_{i_1,\ldots,i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}, \quad \bar{\Delta}\alpha = 0 \quad \text{and} \quad \delta_0 \alpha = 0,$$

where $\alpha_{i_1,\ldots,i_q}$ are homogeneous polynomials of degree k.

Denoting by $i : S^n \to \mathbb{R}^{n+1}$ the inclusion map, it is proved on [7, Corollary 6.6] that for every $q \in \{0\} \cup \mathbb{N}_n$, the restriction of the pullback function $i^* : \sum_{k\geq 0} H_k^q \to C^\infty(\Lambda^q(S^n))$ is injective and has its image dense in $C^\infty(\Lambda^q(S^n))$.

Considering the natural actions of $SO(n+1)$ on $C^\infty(\Lambda^q(\mathbb{R}^{n+1}))$ and $C^\infty(\Lambda^q(S^n))$ we see that H_k^q and $i^*H_k^q$ are $SO(n+1)$-modules, and moreover, i^* commutes with these actions. Therefore, by decomposing H_k^q into $SO(n+1)$-irreducible modules one obtains the corresponding decomposition of $i^*H_k^q$. This decomposition was established by Ikeda and Taniguchi on [7, Theorem 6.8] and is presented in the odd dimensional case on Table 1, which uses the same roots adopted by Weng and You on [15], namely, $\prod = \{\pm e_i \pm e_j \mid i \neq j\}$ is the root system, $\prod_+ = \{e_i - e_j, e_i + e_j \mid i < j\}$ are the positive roots and $\{e_1 - e_2, e_2 - e_3, \ldots, e_{m-1} - e_m, e_{m-1} + e_m\}$ are the simple roots, as opposed to the roots adopted on [7], with the convention that for $1 \leqslant q \leqslant m$, $E(-1\Lambda_1 + \Lambda_q) = E(-1\Lambda_1 + \Lambda_m^-) = E(-1\Lambda_1 + \Lambda_m^+) = \{0\}$ is the trivial module.

Table 1: Decomposition of the spaces $i^*H_k^q$ into $SO(n+1)$-irreducible modules. These data were first presented at [7].

$n = 2m-1,\ m \geqslant 1$	
q	H_k^q
$q = 0$	$E(k\Lambda_1)$
$1 \leqslant q \leqslant m-2$	$E(k\Lambda_1 + \Lambda_q) \oplus E((k-1)\Lambda_1 + \Lambda_{q+1})$
$q = m-1$	$E(k\Lambda_1 + \Lambda_{m-1}) \oplus E((k-1)\Lambda_1 + \Lambda_m^-) \oplus E((k-1)\Lambda_1 + \Lambda_m^+)$
$q = m$	$E((k-1)\Lambda_1 + \Lambda_{m-1}) \oplus E(k\Lambda_1 + \Lambda_m^-) \oplus E(k\Lambda_1 + \Lambda_m^+)$
$m+1 \leqslant q \leqslant 2m-2$	$E((k-1)\Lambda_1 + \Lambda_{n-q}) \oplus E(k\Lambda_1 + \Lambda_{n+1-q})$
$q = 2m-1$	$E((k+1)\Lambda_1)$

By using [5, section 3.9], we have that given a compact Riemannian manifold without boundary M whose fundamental group is cyclic of order l, by selecting representatives $(\rho_s)_{0\leqslant s\leqslant l}$ for the irreducible representations of $\pi_1(M)$, it is valid that:

$$E(\lambda, \Delta_{\tilde{M}}^q) = \oplus_s \pi^* E(\lambda, \Delta_{M,\rho_s}^q), \tag{1}$$

where $\pi : \tilde{M} \to M$ is the universal covering map of M, Δ_{M,ρ_s}^q is the twisted Laplacian associated to ρ_s and $\pi^*E(\lambda, \Delta_{M,\rho_s}^q)$ is the space of equivariant eigenforms of $\Delta_{\tilde{M}}^q$ associated to $\lambda \in \mathbb{R}$ and ρ_s.

So, by identifying among the eigenforms of $\Delta_{S^n}^q$, $n \geqslant 2$, the ones that are equivariant in relation to the trivial representation σ of $\pi_1(\mathbb{R}\mathrm{P}^n)$ in $U(1)$ we obtain the eigenvalues and dimension of the eigenspaces of $\Delta_{\mathbb{R}\mathrm{P}^n}^q \simeq \Delta_{\mathbb{R}\mathrm{P}^n,\sigma}^q$.

These eigenforms are precisely the pullbacks by the inclusion map $i : S^n \to \mathbb{R}^{n+1}$ of the harmonic polynomial forms $\omega \in C^\infty(\Lambda^q\mathbb{R}^{n+1})$ that satisfy $\omega(x) = \omega(-x)$ for every $x \in \mathbb{R}^{n+1}$. If H_k^q is one space of harmonic polynomial forms, then H_k^q contains forms with this property if, and only if, $q+k$ is even and in this case every element of H_k^q has this property.

Therefore, we may apply the results of Table 1 and (1) to decompose $C^\infty(\Lambda^q\mathbb{R}\mathrm{P}^n)$, $n \geqslant 2$, n odd, into irreducible $SO(n+1)$-modules, as presented in Table 2. In the case $n=1$ we cannot apply the theory of [5, section 3.9] to study the spectrum of $\mathbb{R}\mathrm{P}^1$, since $\pi_1(\mathbb{R}\mathrm{P}^1) \simeq \mathbb{Z}$ and S^1 is not the universal cover of this space. Nevertheless, the spectrum and eigenspaces of $\Delta^0_{\mathbb{R}\mathrm{P}^1}$ are known, see [3, Section 3.2.B], and by Hodge duality, for an arbitrary orientable manifold M of dimension n we have for every $q \in \mathbb{N}_n \cup \{0\}$ that Δ^q_M and Δ^{n-q}_M have the same eigenvalues and the corresponding eigenspaces are isomorphic. So, the results that would be obtained if we applied the theory of [5, section 3.9] in this case, considering the natural 2-fold Riemannian covering map of S^1 on $\mathbb{R}\mathrm{P}^1$, are correct. Table 2 summarizes the decomposition of $C^\infty(\Lambda^q\mathbb{R}\mathrm{P}^n)$, $n \geqslant 2$, n odd and the decomposition of $C^\infty(\Lambda^q\mathbb{R}\mathrm{P}^1)$ into irreducible $SO(n+1)$-modules.

Table 2: Decomposition of the spaces of forms into $SO(n+1)$-irreducible modules on odd dimensional projective spaces

$n=2m-1,\ m\geqslant 1$	
q	$C^\infty(\Lambda^q\mathbb{R}\mathrm{P}^n)$
$q=0$	$\bigoplus_{k\geqslant 0} E(2k\Lambda_1)$
$1\leqslant q\leqslant m-2$; q even	$\bigoplus_{k\geqslant 0} E(2k\Lambda_1+\Lambda_q)\oplus E((2k+1)\Lambda_1+\Lambda_{q+1})$
$1\leqslant q\leqslant m-2$; q odd	$\bigoplus_{k\geqslant 0} E((2k+1)\Lambda_1+\Lambda_q)\oplus E(2k\Lambda_1+\Lambda_{q+1})$
$q\in\{m-1,m\}$; m even	$\bigoplus_{k\geqslant 0} E((2k+1)\Lambda_1+\Lambda_{m-1})\oplus E(2k\Lambda_1+\Lambda_m^-)\oplus E(2k\Lambda_1+\Lambda_m^+)$
$q\in\{m-1,m\}$; m odd	$\bigoplus_{k\geqslant 0} E(2k\Lambda_1+\Lambda_{m-1})\oplus E((2k+1)\Lambda_1+\Lambda_m^-)\oplus E((2k+1)\Lambda_1+\Lambda_m^+)$
$m+1\leqslant q\leqslant 2m-2$; q even	$\bigoplus_{k\geqslant 0} E((2k+1)\Lambda_1+\Lambda_{n-q})\oplus E(2k\Lambda_1+\Lambda_{n+1-q})$
$m+1\leqslant q\leqslant 2m-2$; q odd	$\bigoplus_{k\geqslant 0} E(2k\Lambda_1+\Lambda_{n-q})\oplus E((2k+1)\Lambda_1+\Lambda_{n+1-q})$
$q=2m-1$	$\bigoplus_{k\geqslant 0} E(2k\Lambda_1)$

On Table 3 we present the weights, corresponding eigenvalue and dimension of the associated spaces. This table summarizes part of the data present on [15, section 2], but we correct a misprint of these data which corresponds to the last line, last column of the table for $q \neq m$. There is a proof for the misprint of Weng and You data on the appendix of [12].

Table 3: Associated eigenvalues and dimensions of the $SO(n+1)$-irreducible modules

Weights	c_Λ	$\dim E(\Lambda)$
$k\Lambda_1$	$k(k+n-1)$	$(2k+2m-2)\frac{(k+2m-3)!}{k!(2m-2)!}$
$k\Lambda_1+\Lambda_q$, $2\leqslant q\leqslant m-1$	$(k+q)(k+n+1-q)$	$\frac{2(k+m)(k+2m-1)}{(k+q)(k+2m-q)}\binom{k+2m-2}{k,q-1,2m-q-1}$
$k\Lambda_1+\Lambda_m^\pm$	$(k+m)^2$	$\binom{k+m-1}{k}\binom{k+2m-1}{m-1}$

Finally, taking into account the fact that every unitary representation of a finite group may be decomposed into a direct sum of irreducible representations; the fact that

$\pi_1(\mathbb{R}\mathrm{P}^{2m-1})$ has only two irreducible representations, the trivial σ_1 and the non-trivial σ_2; and the expression (1), we establish Table 4 with the eigenvalues and dimension of the eigenspaces of the twisted Laplacians $\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\rho}$ in relation to the representation

$$\rho \simeq \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{\gamma} \oplus \underbrace{\sigma_2 \oplus \cdots \oplus \sigma_2}_{N-\gamma}.$$

On this table we use the function E^k_q, which is defined for $k,q \in \mathbb{N}\cup\{0\}$ by $E^k_q = \gamma$, if $k+q \equiv 0 \mod 2$ and $E^k_q = N-\gamma$, if $k+q \equiv 1 \mod 2$.

Table 4: Eigenvalues and dimension of the eigenspaces of the twisted Laplacian $\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}$. The value of k varies in $\{0\}\cup\mathbb{N}$.

degree	eigenvalue	dimension of the eigenspace
0	$k(k+2m-2)$	$E^k_0 \cdot (2k+2m-2)\,\frac{(k+2m-3)!}{k!(2m-2)!}$
$1 \leq q \leq m-1$	$(k+q)(k+2m-q)$	$E^k_q \cdot \frac{2(k+m)(k+2m-1)}{(k+q)(k+2m-q)} \binom{k+2m-2}{k,q-1,2m-q-1}$
$1 \leq q \leq m-1$	$(k+q+1)(k+2m-q-1)$	$E^k_{q+1} \cdot \frac{2(k+m)(k+2m-1)}{(k+q+1)(k+2m-q-1)} \binom{k+2m-2}{k,q,2m-q-2}$
$m \leq q \leq 2m-2$	$(k+2m-q)(k+q)$	$E^k_q \cdot \frac{2(k+m)(k+2m-1)}{(k+2m-q)(k+q)} \binom{k+2m-2}{k,2m-1-q,q-1}$
$m \leq q \leq 2m-2$	$(k+2m-q-1)(k+q+1)$	$E^k_{q+1} \cdot \frac{2(k+m)(k+2m-1)}{(k+2m-q-1)(k+q+1)} \binom{k+2m-2}{k,2m-q-2,q}$
$2m-1$	$k(k+2m-2)$	$E^k_0 \cdot (2k+2m-2)\,\frac{(k+2m-3)!}{k!(2m-2)!}$

3 Determinants on odd dimensional projective spaces

We start now the construction of the explicit formulae for determinants. By using Table 3 we may define for $q \in \mathbb{N}_{m-1}$ the auxiliary functions:

$$\begin{array}{lcll}
\xi^o_{0,k}(s) & = & \dim E(k\Lambda_1) c^{-s}_{k\Lambda_1}, & k \in \mathbb{N}, \\
\xi^o_{q,k}(s) & = & \dim E(k\Lambda_1+\Lambda_q) c^{-s}_{k\Lambda_1+\Lambda_q}, & k \in \mathbb{N}\cup\{0\}, \\
\xi^{o,-}_{m,k}(s) & = & \dim E(k\Lambda_1+\Lambda^-_m) c^{-s}_{k\Lambda_1+\Lambda^-_m}, & k \in \mathbb{N}\cup\{0\}, \\
\xi^{o,+}_{m,k}(s) & = & \dim E(k\Lambda_1+\Lambda^+_m) c^{-s}_{k\Lambda_1+\Lambda^+_m}, & k \in \mathbb{N}\cup\{0\}, \\
\xi^o_{m,k}(s) & = & \xi^{o,-}_{m,k}(s)+\xi^{o,+}_{m,k}(s), & k \in \mathbb{N}\cup\{0\}.
\end{array} \tag{2}$$

These functions are the same auxiliary functions defined in [12, section 3.1] to calculate the regularized determinants of odd dimensional spheres. By using them and the decomposition of $C^\infty(\Lambda^q\mathbb{R}\mathrm{P}^{2m-1})$ into $SO(2m)$-irreducible modules given by table 2, remembering to exclude the eigenvalue 0 on $\zeta(s,\Delta^0_{\mathbb{R}\mathrm{P}^{2m-1}})$ and $\zeta(s,\Delta^{2m-1}_{\mathbb{R}\mathrm{P}^{2m-1}})$, we may describe the spectral zeta functions of $\mathbb{R}\mathrm{P}^{2m-1}$ in the following way:

$$\begin{array}{lcll}
\zeta(s,\Delta^0_{\mathbb{R}\mathrm{P}^{2m-1}}) & = & \sum\limits_{k\geqslant 1} \xi^o_{0,2k}(s) = \sum\limits_{k\geqslant 0} \xi^o_{1,2k+1}(s) & \\
\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}) & = & \sum\limits_{k\geqslant 0} \xi^o_{q,2k}(s) + \xi^o_{q+1,2k+1}(s); & 1 \leqslant q \leqslant m-1, \quad q \quad \text{even} \\
\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}) & = & \sum\limits_{k\geqslant 0} \xi^o_{q,2k+1}(s) + \xi^o_{q+1,2k}(s); & 1 \leqslant q \leqslant m-1, \quad q \quad \text{odd} \\
\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}) & = & \zeta(s,\Delta^{2m-1-q}_{\mathbb{R}\mathrm{P}^{2m-1}}); & m \leqslant q \leqslant 2m-1.
\end{array}$$

So, to determine the derivatives at zero of the spectral zeta functions we only need to calculate the derivatives at zero of the functions $\sum\limits_{k\geqslant 0}\xi^{o}_{2a,2k}(s)$ and $\sum\limits_{k\geqslant 0}\xi^{o}_{2a+1,2k+1}(s)$ for $2a$, $2a+1\in\mathbb{N}_m$.

By defining $u^q_l := (l-q)(2m-q-l)$ and $u^q := (u^q_l)_{l\in\mathbb{N}_m-\{q\}} = (u^q_1,\ldots,u^q_{q-1},u^q_{q+1},\ldots,u^q_m)\in\mathbb{Z}^{m-1}$, and using Lemma 6.5 with $q=2a$ in the first equality below and part S_1 of Lemma 6.4 with $n=2m-1$ in the last one, we have for every $a\in\mathbb{N}$ such that $2a\in\mathbb{N}_m$:

$$\begin{aligned}
\sum_{k\geqslant 0}\xi^{o}_{2a,2k}(s) &= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant 0}\frac{1}{(2k+2a)^{s-j}(2k+2m-2a)^{s-j}}\\
&= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant a}\frac{1}{(2k)^{s-j}(2k+2m-4a)^{s-j}}\\
&= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant 1}\frac{1}{(2k)^{s-j}(2k+2m-2q)^{s-j}}\\
&- \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k=1}^{a-1}\frac{1}{(2k)^{s-j}(2k+2m-2q)^{s-j}}\\
&= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant 1}\frac{1}{(2k)^{s-j}(2k+2m-2q)^{s-j}}\\
&- \sum_{k=1}^{a-1}\frac{2}{(2m-q-1)!(q-1)!(2k)^s(2k+2m-2q)^s}\sum_{j=0}^{m-1}e_{m-j-1}(u^q)(2k)^j(2k+2m-2q)^j\\
&= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant 1}\frac{1}{(2k)^{s-j}(2k+2m-2q)^{s-j}},
\end{aligned}$$

where $e_{m-j-1}(u^q)$ denote elementary symmetric polynomials on u^q:

$$\prod_{l=1}^{m}\left(x+u^q_l\right) = \sum_{j=0}^{m-1}e_{m-j-1}(u^q)x^j.$$

Using the same Lemmas with $q=2a+1$ we have for every $a\in\mathbb{N}$ such that $2a+1\in\mathbb{N}_m$:

$$\begin{aligned}
\sum_{k\geqslant 0}\xi_{2a+1,2k+1}(s) &= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant 0}\frac{1}{(2k+2a+2)^{s-j}(2k+2m-2a)^{s-j}}\\
&= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant a+1}\frac{1}{(2k)^{s-j}(2k+2m-4a-2)^{s-j}}\\
&= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant 1}\frac{1}{(2k)^{s-j}(2k+2m-2q)^{s-j}}\\
&- \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k=1}^{a}\frac{1}{(2k)^{s-j}(2k+2m-2q)^{s-j}}\\
&= \sum_{j=0}^{m-1}\frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k\geqslant 1}\frac{1}{(2k)^{s-j}(2k+2m-2q)^{s-j}}.
\end{aligned}$$

So the problem of calculating $\left(\sum\limits_{k\geqslant 0}\xi_{2a,2k}(s)\right)'\Big|_{s=0}$ and $\left(\sum\limits_{k\geqslant 0}\xi_{2a+1,2k+1}(s)\right)'\Big|_{s=0}$ is reduced to calculating $\left(\sum\limits_{k\geqslant 1}\frac{1}{(2k)^{s-j}(2k+2\alpha)^{s-j}}\right)'\Big|_{s=0}$ for $\alpha=m-q$. The following lemma is a generalization of [15, Lemma 3.1] and will help us to calculate the desired derivative. Notice that Lemma 6.2 guarantees us that the function $\zeta_{\alpha,\beta,j}(s)$ defined below is well defined and is holomorphic at $s=0$.

Lemma 3.1. *For $\alpha,\ j\in\mathbb{N}\cup\{0\}$ and $\beta\in(0,+\infty)$, the function $\zeta_{\alpha,\beta,j}(s)$ defined by*

$$\zeta_{\alpha,\beta,j}(s)=\sum_{k\geqslant 1}\frac{1}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}},$$

for $\mathrm{Re}(s)\gg 0$ and extended meromorphically to $\mathbb{C}$ satisfies:

$$\zeta'_{\alpha,\beta,j}(0)=\sum_{k=1}^{\alpha}\log(\beta k)(\beta k)^j(\beta(k-\alpha))^j \\ +\sum_{l}\binom{j}{2l}(\beta\alpha)^{2l}\left(2^{-2t+2j-2l}\zeta_R(2t-2j+2l)\right)'\Big|_{t=0}.$$

Proof. Differentiating the function $\zeta_{\alpha,\beta,j}(s)$ term by term we obtain:

$$\begin{aligned}
\zeta'_{\alpha,\beta,j}(s) &= -\sum_{k\geqslant 1}\frac{\log(\beta k)}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}}-\sum_{k\geqslant 1}\frac{\log(\beta k+\beta\alpha)}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}}\\
&= -\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}}-\sum_{k\geqslant\alpha+1}\frac{\log(\beta k)}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}}\\
&\quad -\sum_{k\geqslant\alpha+1}\frac{\log(\beta k)}{(\beta k-\beta\alpha)^{s-j}(\beta k)^{s-j}}\\
&= -\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}}-\sum_{k\geqslant\alpha+1}\frac{\log(\beta k)}{(\beta k)^{2s-2j}}\left(1+\frac{\beta\alpha}{\beta k}\right)^{-s+j}\\
&\quad -\sum_{k\geqslant\alpha+1}\frac{\log(\beta k)}{(\beta k)^{2s-2j}}\left(1-\frac{\beta\alpha}{\beta k}\right)^{-s+j}
\end{aligned}$$

Using $\left|\frac{\alpha}{k}\right|<1$ for $k\geqslant\alpha+1$ and the Binomial Theorem we have:

$$\begin{aligned}
\zeta'_{\alpha,\beta,j}(s) &= -\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}}-\sum_{k\geqslant\alpha+1}\frac{\log(\beta k)}{(\beta k)^{2s-2j}}\sum_{l}\binom{-s+j}{l}\left(\frac{\beta\alpha}{\beta k}\right)^{l}\\
&\quad -\sum_{k\geqslant\alpha+1}\frac{\log(\beta k)}{(\beta k)^{2s-2j}}\sum_{l}\binom{-s+j}{l}\left(-\frac{\beta\alpha}{\beta k}\right)^{l}\\
&= -\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}}\\
&\quad -\sum_{l}\binom{-s+j}{l}(\beta\alpha)^{l}\left[\sum_{k\geqslant 1}\frac{\log(\beta k)}{(\beta k)^{2s-2j+l}}-\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{2s-2j+l}}\right]\\
&\quad -\sum_{l}\binom{-s+j}{l}(-\beta\alpha)^{l}\left[\sum_{k\geqslant 1}\frac{\log(\beta k)}{(\beta k)^{2s-2j+l}}-\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{2s-2j+l}}\right]\\
&= -\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{s-j}(\beta k+\beta\alpha)^{s-j}}+\sum_{l}\binom{-s+j}{l}\Bigg[(\beta\alpha)^{l}\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{2s-2j+l}}\\
&\quad +(-\beta\alpha)^{l}\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{2s-2j+l}}+(\beta\alpha)^{l}\frac{1}{2}(\beta^{-2s+2j-l}\zeta_R(2s-2j+l))'\\
&\quad +(-\beta\alpha)^{l}\frac{1}{2}(\beta^{-2s+2j-l}\zeta_R(2s-2j+l))'\Bigg].
\end{aligned} \tag{3}$$

The Riemann zeta function has a simple pole of residuum 1 in $s=1$ and is holomorphic in the rest of the complex plane. So the function $(\beta^{-2s+2j-l}\zeta_R(2s-2j+l))'$ has a pole of order 2 in $s=j-\frac{l}{2}+\frac{1}{2}$ and is holomorphic on $\mathbb{C}-\{j-\frac{l}{2}+\frac{1}{2}\}$. This pole happens at $s=0$ if, and only if, $l=2j+1$ and in this case we have:

$$(\beta\alpha)^l\frac{1}{2}(\beta^{-2s+2j-l}\zeta_R(2s-2j+l))'+(-\beta\alpha)^l\frac{1}{2}(\beta^{-2s+2j-l}\zeta_R(2s-2j+l))'=0.$$

Therefore, evaluating expression (3) at $s=0$, we obtain:

$$\begin{aligned}
\zeta'_{\alpha,\beta,j}(0) &= -\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{-j}(\beta k+\beta\alpha)^{-j}}+\sum_{l}\begin{pmatrix} j\\ 2l\end{pmatrix}(\beta\alpha)^{2l}(\beta^{-2s+2j-2l}\zeta_R(2s-2j+2l))'|_{s=0}\\
&+ \sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{-2j}}\sum_{l}\begin{pmatrix} j\\ l\end{pmatrix}\left(\frac{\beta\alpha}{\beta k}\right)^{l}+\sum_{k=1}^{\alpha}\frac{\log(\beta k)}{(\beta k)^{-2j}}\sum_{l}\begin{pmatrix} j\\ l\end{pmatrix}\left(-\frac{\beta\alpha}{\beta k}\right)^{l}\\
&= \sum_{k=1}^{\alpha}\log(\beta k)(\beta k)^j(\beta k-\beta\alpha)^j+\sum_{l}\begin{pmatrix} j\\ 2l\end{pmatrix}(\beta\alpha)^{2l}(\beta^{-2s+2j-2l}\zeta_R(2s-2j+2l))'|_{s=0}
\end{aligned}$$

□

By using Lemma 3.1 with $\alpha=m-q$ and $\beta=2$ we have for $\phi\in\{0,1\}$ and $q=2a+\phi$:

$$\left(\sum_{k\geqslant 0}\xi_{2a+\phi,2k+\phi}(s)\right)'\Bigg|_{s=0}=2\sum_{j=0}^{m-1}\frac{e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\left[\sum_{k=1}^{m-q}\log(2k)(2k)^j(2k-2m+2q)^j\right.$$
$$\left.+\sum_{l}\begin{pmatrix} j\\ 2l\end{pmatrix}(2m-2q)^{2l}\left(2^{-2t+2j-2l}\zeta_R(2t-2j+2l)\right)'\Big|_{t=0}\right].$$

By defining

$$\begin{aligned}
Z_1^q &= 2\sum_{j=0}^{m-1}\frac{e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k=1}^{m-q}(2k)^j(2k-2m+2q)^j\log 2k,\\
Z_2^q &= 2\sum_{j=0}^{m-1}\frac{e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{l}\begin{pmatrix} j\\ 2l\end{pmatrix}(2m-2q)^{2l}\left(2^{-2t+2j-2l}\zeta_R(2t-2j+2l)\right)'\Big|_{t=0},
\end{aligned}$$

we obtain $\left(\sum_{k\geqslant 0}\xi_{2a+\phi,2k+\phi}(s)\right)'\Big|_{s=0}=Z_1^{2a+\phi}+Z_2^{2a+\phi}$.

For $q=m$ we have $Z_1^m=0$ by definition and for $q\in\mathbb{N}_{m-1}$, by applying part S_2 of Lemma 6.4 in the fourth equality below we have:

$$\begin{aligned}
Z_1^q &= 2\sum_{j=0}^{m-1}\frac{e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!}\sum_{k=1}^{m-q}\log(2k)(2k)^j(2k-2m+2q)^j\\
&= 2\sum_{k=1}^{m-q}\frac{\log(2k)}{(2m-q-1)!(q-1)!}\sum_{j=0}^{m-1}e_{m-j-1}(u^q)(2k)^j(2k-2m+2q)^j\\
&= 2\sum_{k=1}^{m-q}\frac{\log(2k)}{(2m-q-1)!(q-1)!}\prod_{j=1,j\neq q}^{m}(2k(2k-2m+2q)+u_j^q)\\
&= \frac{2\log(2m-2q)}{(2m-q-1)!(q-1)!}\prod_{j=1,j\neq q}^{m}u_j^q=\frac{2\log(2m-2q)}{(2m-q-1)!(q-1)!}\prod_{j=1,j\neq q}^{m}(j-q)(2m-q-j)\\
&= \frac{2\log(2m-2q)}{(2m-q-1)!(q-1)!}\frac{(-1)^{q-1}(q-1)!(2m-q-1)!}{2}\\
&= (-1)^{q-1}\log(2m-2q).
\end{aligned}$$

About Z_2^q, substituting $(2^{-2t+2j-2l}\zeta_R(2t-2j+2l))'|_{t=0}$ by x^{2j-2l} we obtain half of the polynomial expression of Lemma 6.6, so we may conclude that

$$Z_2^q = \frac{2}{(2m-q-1)!(q-1)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_1^q)\left(2^{-2t+2k}\zeta_R(2t-2k)\right)'\Big|_{t=0} \tag{4}$$

for

$$\begin{array}{rcl} w_1^q & = & (l-q)_{l\in\mathbb{N}_m-\{q\}}\times(2m-l-q)_{l\in\mathbb{N}_m-\{q\}} \\ & = & (1-q,\ldots,\widehat{q-q},\ldots m-q,2m-1-q,\ldots,\widehat{2m-2q},\ldots m-q) \end{array},$$

and $e_{2m-2k-2}(w_1^q)$ the elementary symmetric polynomials in w_1^q.

By using

$$\forall k\in\mathbb{N}:\quad \zeta_R(-2k)=0,\quad \zeta_R(0)=-\frac{1}{2}, \tag{5}$$

and the expression

$$e_{2m-2}(w_1^q)=\begin{cases} 1, & \text{if } m=1 \text{ and consequently } q=1,\\ (-1)^{q-1}\frac{(q-1)!(2m-q-1)!}{2}, & \text{if } m>1,\end{cases} \tag{6}$$

We may write Z_2^q as:

$$Z_2^q=\tfrac{4}{(2m-q-1)!(q-1)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_1^q)2^k\zeta_R'(-2k)+(-1)^{q-1}(1+\delta_m^1)\log 2.$$

Since $\left(\sum_{k\geqslant 0}\xi_{2a+\phi,2k+\phi}(s)\right)'\Big|_{s=0}=Z_1^{2a+\phi}+Z_2^{2a+\phi}$, we have the following Theorem as conclusion.

Theorem 3.2. *The derivatives at zero of the spectral zeta functions on odd dimensional projective spaces are given by: ($m\geqslant 1$)*

$$\begin{array}{rcl} \zeta'(0,\Delta^0_{\mathbb{RP}^{2m-1}}) & = & (1+\delta_m^1)\log 2+\log(2m-2+\delta_m^1) \\ & + & \tfrac{4}{(2m-2)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_1^1)2^{2k}\zeta_R'(-2k) \\ \zeta'(0,\Delta^q_{\mathbb{RP}^{2m-1}}) & = & (-1)^{q-1}\log(2m-2q)+(-1)^q\log(2m-2q-2+\delta^q_{m-1}) \\ & + & \tfrac{4}{(2m-q-1)!(q-1)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_1^q)2^{2k}\zeta_R'(-2k) \\ & + & \tfrac{4}{(2m-q-2)!q!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_1^{q+1})2^{2k}\zeta_R'(-2k);\ 1\leqslant q\leqslant m-1 \\ \zeta'(0,\Delta^q_{\mathbb{RP}^{2m-1}}) & = & \zeta'(0,\Delta^{2m-1-q}_{\mathbb{RP}^{2m-1}});\quad m\leqslant q\leqslant 2m-1 \end{array}$$

where

$$w_1^q=(l-q)_{l\in\mathbb{N}_m-\{q\}}\times(2m-l-q)_{l\in\mathbb{N}_m-\{q\}}.$$

and $e_{2m-2k-2}(w_1^1)$ are elementary symmetric polynomials.

It is interesting to note that Theorem 3.2 of this text and Theorem 4.1 of [12] imply that regularized determinants are not diffeomorphism invariants, since S^1 is diffeomorphic to $\mathbb{R}\mathrm{P}^1$ and

$$\det\Delta^0_{S^1} = \det\Delta^1_{S^1} \neq \det\Delta^1_{\mathbb{R}\mathrm{P}^1} = \det\Delta^0_{\mathbb{R}\mathrm{P}^1}.$$

This fact is also observable from the definition of $\zeta(s,\Delta^p_M)$ and the spectral data of S^1 and $\mathbb{R}\mathrm{P}^1$, since

$$\zeta(s,\Delta^0_{S^1}) = 2\zeta_R(2s) \quad \text{and} \quad \zeta(s,\Delta^0_{\mathbb{R}\mathrm{P}^1}) = 2^{-2s+1}\zeta_R(2s).$$

Theorem 3.2 may also be used alongside Theorem 4.1 of [12] and expression (1) or lemma 6.7 to describe the regularized determinant of the twisted Laplacian $\Delta^p_{\mathbb{R}\mathrm{P}^{2m-1},\rho}$ associated to a unitary representation ρ of the fundamental group of $\mathbb{R}\mathrm{P}^{2m-1}$ in terms of the determinant of $\Delta^p_{\mathbb{R}\mathrm{P}^{2m-1}}$, as described in the next proposition.

Theorem 3.3. *Let $\rho : G \to U(N)$ be a unitary representation of the deck transformation group G of the 2-fold Riemannian covering of S^{2m-1} on $\mathbb{R}\mathrm{P}^{2m-1}$ and let*

$$\rho \simeq \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{\gamma} \oplus \underbrace{\sigma_2 \oplus \cdots \oplus \sigma_2}_{N-\gamma}, \tag{7}$$

be a decomposition of ρ as a direct sum of irreducible representations, where σ_1 and σ_2 are respectively the trivial and non-trivial irreducible representations of $\pi_1(\mathbb{R}\mathrm{P}^{2m-1})$. The regularized determinant of the twisted Laplacian $\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\rho}$ is given in terms of the determinant of $\Delta^q_{S^{2m-1}}$ and $\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}$ by:

$$\det\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\rho} = (\det\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}})^{2\gamma-N}\cdot(\det\Delta^q_{S^{2m-1}})^{N-\gamma}.$$

Proof. By (1), $m \geqslant 2$, or lemma 6.7 for $m = 1$, we have for every $\lambda \in \mathbb{R}$:

$$\dim E(\lambda,\Delta^q_{S^{2m-1}}) = \dim E(\lambda,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\sigma_1}) + \dim(\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\sigma_2}).$$

Since

$$\dim E(\lambda,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\rho}) = \gamma(\dim E(\lambda,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\sigma_1})) + (N-\gamma)(\dim E(\lambda,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\sigma_2})),$$

and

$$\dim E(\lambda,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\sigma_1}) = \dim E(\lambda,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}),$$

it follows from the definition of spectral zeta function that

$$\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1},\rho}) = (2\gamma-N)\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}) + (N-\gamma)\zeta(s,\Delta^q_{S^{2m-1}}).$$

The result follows from this expression and the definition of regularized determinant. □

4 Equivalence with the previous formula

In this section we prove the equivalence between our formula for $\det\Delta^0_{\mathbb{R}P^{2m-1}}$ and the one established by Hartmann-Spreafico for the regularized determinant of the Laplacian on odd dimensional real projective spaces.

The formula obtained by these authors in this case is given by:

$$\begin{aligned}
\zeta'(0,\Delta^0_{\mathbb{R}P^{2m-1}}) &= \frac{4}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\sum_{j=0}^{l}\begin{pmatrix}2l\\2j\end{pmatrix}(m-1)^{2j}2^{2l-2j}\zeta_R'(2j-2l)\\
&+ \frac{2}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\sum_{j=0}^{2l}\begin{pmatrix}2l\\j\end{pmatrix}(-1)^j(m-1)^j\sum_{t=1}^{m-1}(2t)^{2l-j}\log t\\
&- \frac{4}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)(m-1)^{2l}\zeta_R(0)\log 2\\
&+ \frac{2}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\left[-2\sum_{j=0}^{l-1}\begin{pmatrix}2l\\2j+1\end{pmatrix}(m-1)^{2j+1}\right.\\
&\times \left. 2^{2l-2j-1}\zeta_R(2j+1-2l)+\frac{(m-1)^{2l+1}}{2l+1}\right]\log 2,
\end{aligned} \tag{8}$$

where $d^0=(-(m-l-1)^2)_{l\in\mathbb{N}_{m-1}}$.

On the other hand, the formula obtained by us in this case is given by:

$$\begin{aligned}
\zeta'(0,\Delta^0_{\mathbb{R}P^{2m-1}}) &= \log(m-1+\delta^1_m)+2\log 2\\
&+ \frac{4}{(2m-2)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w^1_1)2^{2k}\zeta_R'(-2k),
\end{aligned}$$

for $w^1_1=(l-1)_{l\in\mathbb{N}_m-\{1\}}\times(2m-l-1)_{l\in\mathbb{N}_m-\{1\}}$.

Analysing the first line of (8), we may substitute $2^{2l-2j}\zeta_R'(2j-2l)$ by x^{2l-2j}, so that the corresponding expression becomes

$$\begin{aligned}
&\frac{4}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\sum_{j=0}^{l}\begin{pmatrix}2l\\2j\end{pmatrix}(m-1)^{2j}x^{2l-2j}\\
=\;&\frac{2}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\sum_{j=0}^{2l}\left[\begin{pmatrix}2l\\j\end{pmatrix}(m-1)^j x^{2l-j}+\begin{pmatrix}2l\\j\end{pmatrix}(-1)^j(m-1)^j x^{2l-j}\right]\\
=\;&\frac{2}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\left[((x+m-1)^2)^l+((x-m+1)^2)^l\right].
\end{aligned} \tag{9}$$

Since

$$\begin{aligned}
\sum_{l=0}^{m-1} e_{m-1-l}(d^0)((x+m-1)^2)^l &= \prod_{l=1}^{m-1}((x+m-1)^2-(m-l-1)^2)\\
= \prod_{l=1}^{m-1}(x+2m-2-l)(x+l) &= \prod_{l=2}^{m}(x+2m-1-l)(x+l-1),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{l=0}^{m-1} e_{m-1-l}(d^0)((x-m+1)^2)^l &= \prod_{l=1}^{m-1}((x-m+1)^2-(m-l-1)^2)\\
= \prod_{l=1}^{m-1}(x-2m+2+l)(x-l) &= \prod_{l=2}^{m}(x-2m+1+l)(x-l+1),
\end{aligned}$$

it follows by Lemma 6.6 that (9) is equal to $\frac{4}{(2m-2)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_1^1)x^{2k}$, and substituting back x^{2k} by $2^{2k}\zeta_R'(-2k)$, we obtain the following equivalent expression for the first line of (8):

$$\frac{4}{(2m-2)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_1^1)2^{2k}\zeta_R'(-2k).$$

About the second line, it is equal to 0 for $m=1$, and for $m \geqslant 1$ we have:

$$\begin{aligned}
& \tfrac{2}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\sum_{j=0}^{2l}\begin{pmatrix} 2l \\ j \end{pmatrix}(-1)^j(m-1)^j\sum_{t=1}^{m-1}(2t)^{2l-j}\log t \\
= \; & \tfrac{2}{(2m-2)!}\sum_{t=1}^{m-1}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)(2t)^{2l}\log t\sum_{j=0}^{2l}\begin{pmatrix} 2l \\ j \end{pmatrix}(-1)^j(m-1)^j(2t)^{-j} \\
= \; & \tfrac{2}{(2m-2)!}\sum_{t=1}^{m-1}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)(2t)^{2l}\log t\left(1+\tfrac{1-m}{2t}\right)^{2l} \\
= \; & \tfrac{2}{(2m-2)!}\sum_{t=1}^{m-1}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\log t\,(2t+1-m)^{2l} \\
= \; & \tfrac{2}{(2m-2)!}\sum_{t=1}^{m-1}\log t\prod_{l=1}^{m-1}(-(m-1-l)^2+(m-1-2t)^2) \\
= \; & \tfrac{2}{(2m-2)!}\log(m-1)\prod_{l=1}^{m-1}(-(m-1-l)^2+(m-1)^2) \\
= \; & \tfrac{2}{(2m-2)!}\log(m-1)\prod_{l=1}^{m-1} l(2m-2-l) \\
= \; & \log(m-1).
\end{aligned}$$

In relation to the third line of (8), this expression is not well defined for $m=1$ since it requires the calculus of 0^0, and for $m>1$ we have:

$$\begin{aligned}
& -\tfrac{4}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)(m-1)^{2l}\zeta_R(0)\log 2 \\
= \; & \tfrac{2\log 2}{(2m-2)!}\sum_{l=0}^{m-1} e_{m-1-l}(d^0)(m-1)^{2l} \\
= \; & \tfrac{2\log 2}{(2m-2)!}\prod_{l=1}^{m-1}((m-1)^2-(m-l-1)^2) = \tfrac{2\log 2}{(2m-2)!}\prod_{l=1}^{m-1} l(2m-2-l) = \log 2.
\end{aligned}$$

About the fourth and fifth lines of (8), the study of these expressions led us to interesting identities involving special values of Bernoulli polynomials and the Riemann zeta function, as described in the following Theorems:

Theorem 4.1. *The Bernoulli polynomials $B_n(x)$ satisfy:*

$$\forall m,l \in \mathbb{N}: \quad B_{2l+1}\left(\frac{m-1}{2}\right) = \sum_{k=0}^{m-3}\left(k^{2l}+(-1)^{k+m+1}k^{2l}\right)\cdot\frac{2l+1}{2^{2l+1}}.$$

Proof. For $m \in \{1,2\}$ we have:

$$\forall l \in \mathbb{N}: \quad B_{2l+1}\left(\frac{1-1}{2}\right) = B_{2l+1} = 0 \quad \text{and} \quad B_{2l+1}\left(\frac{2-1}{2}\right) = \left(2^{1-(2l+1)}-1\right)B_{2l+1} = 0.$$

Assume the result is valid for a certain $m \in \mathbb{N}$. In this case we have for all $l \in \mathbb{N}$ that:

$$\begin{aligned} B_{2l+1}\left(\frac{m+2-1}{2}\right) &= B_{2l+1}\left(\frac{m-1}{2}+1\right) = B_{2l+1}\left(\frac{m-1}{2}\right) + (2l+1)\left(\frac{m-1}{2}\right)^{2l} \\ &= \sum_{k=0}^{m-3}\left(k^{2l}+(-1)^{k+m+1}k^{2l}\right)\cdot\frac{2l+1}{2^{2l+1}} + 2(m-1)^{2l}\cdot\frac{2l+1}{2^{2l+1}} \\ &= \sum_{k=0}^{m-1}\left(k^{2l}+(-1)^{k+m+1}k^{2l}\right)\cdot\frac{2l+1}{2^{2l+1}}. \end{aligned}$$

By induction on $m \in \mathbb{N}$ we have the validity of the Theorem. □

Theorem 4.2. *For all $m \in \mathbb{N}$ the Riemann zeta function satisfies the following identity:*

$$\sum_{l=0}^{m-1} e_{m-1-l}(d^0)\left[-2\sum_{j=0}^{l-1}\binom{2l}{2j+1}(m-1)^{2j+1}2^{2l-2j-1}\zeta_R(2j+1-2l) + \frac{(m-1)^{2l+1}}{2l+1}\right] = \frac{(2m-2)!}{2},$$

where $e_{m-1-l}(d^0)$ are elementary symetric polynomials on $d^0 := \left(-(m-l-1)^2\right)_{l\in\mathbb{N}_{m-1}}$.

Proof. Using the description of the values of the Riemann zeta function on negative integers in terms of Bernoulli numbers, $\zeta_R(-k) = (-1)^k \frac{B_{k+1}}{k+1}$ for $k \in \mathbb{N}$ and $B_r = 0$ for $r \geqslant 3$ odd we have for $l \geqslant 1$:

$$\begin{aligned} &- 2\sum_{j=0}^{l-1}\binom{2l}{2j+1}(m-1)^{2j+1}2^{2l-2j-1}\zeta_R(2j+1-2l) \\ &= 2\sum_{j=0}^{l-1}\binom{2l}{2j+1}(m-1)^{2j+1}2^{2l-2j-1}\tfrac{B_{2l-2j}}{2l-2j} \\ &= 2\sum_{j=0}^{2l-2}\binom{2l}{j+1}(m-1)^{j+1}2^{2l-j-1}\tfrac{B_{2l-j}}{2l-j} \\ &= 2\sum_{j=1}^{2l-1}\binom{2l}{j}(m-1)^{j}2^{2l-j}\tfrac{B_{2l-j+1}}{2l-j+1} \\ &= 2\sum_{j=0}^{2l}\binom{2l}{j}(m-1)^{j}2^{2l-j}\tfrac{B_{2l-j+1}}{2l-j+1} - 2(m-1)^{2l}B_1 \\ &= 2\sum_{j=0}^{2l}\binom{2l}{j}(m-1)^{2l-j}2^{j}\tfrac{B_{j+1}}{j+1} + (m-1)^{2l} \\ &= 2^{2l+1}\sum_{j=0}^{2l}\binom{2l}{j}\left(\tfrac{m-1}{2}\right)^{2l-j}\tfrac{B_{j+1}}{j+1} + (m-1)^{2l}. \end{aligned}$$

Now using Lemma 6.1 with $p = 2l$, $q = 0$, $x = \frac{m-1}{2}$, $y = 0$ and $z = \frac{3-m}{2}$ we have the equality between the last expression above and:

$$-\frac{2^{2l+1}0!(2l)!}{(2l+1)!}\left(\frac{m-1}{2}\right)^{2l+1}-\frac{2^{2l+1}B_{2l+1}\left(\frac{3-m}{2}\right)}{2l+1}+(m-1)^{2l}$$
$$=(m-1)^{2l}\left(1-\frac{m-1}{2l+1}\right)-\frac{2^{2l+1}B_{2l+1}\left(\frac{3-m}{2}\right)}{2l+1}.$$

Since $e_{m-1}(d^0)=0$ and for all $m,l\in\mathbb{N}$, $B_{2l+1}\left(\frac{3-m}{2}\right)=B_{2l+1}\left(\frac{m-1}{2}\right)$ we have by Theorem 4.1

$$\begin{aligned} & \sum_{l=0}^{m-1} e_{m-1-l}(d^0)\left[-2\sum_{j=0}^{l-1}\begin{pmatrix} 2l \\ 2j+1 \end{pmatrix}(m-1)^{2j+1}2^{2l-2j-1}\zeta_R(2j+1-2l)+\frac{(m-1)^{2l+1}}{2l+1}\right] \\ = & \sum_{l=0}^{m-1} e_{m-1-l}(d^0)\left[(m-1)^{2l}-\sum_{k=0}^{m-3}\left(k^{2l}+(-1)^{k+m+1}k^{2l}\right)\right] \\ = & \sum_{l=0}^{m-1} e_{m-1-l}(d^0)(m-1)^{2l}=\left[\prod_{l=1}^{m-1}\left((m-1)^2-(m-1-l)^2\right)\right]=\frac{(2m-2)!}{2} \end{aligned}$$

□

By this Theorem we may conclude the fourth and fifth lines of (8) are equal to $\log 2$ and combining this result with the expressions corresponding to each of the lines of (8) we obtain the equivalence between the formula obtained by Hartmann and Spreafico in [6] and the one obtained by us in this text in the cases covered by both.

5 Analytic Torsion of $\mathbb{R}\mathrm{P}^{2m-1}$

The Analytic Torsion of a closed Riemannian manifold in relation to a representation $\rho:\pi_1(M)\to U(N)$ of its fundamental group is defined in terms of the derivative at zero of the spectral zeta function $\zeta(s,\Delta^q_{M,\rho})$ by:

$$T_a(M,\rho)=\exp\left(\frac{1}{2}\sum_{q=0}^{n}(-1)^q q\zeta'\left(0,\Delta^q_{M,\rho}\right)\right).$$

In this section we use the formulae for regularized determinants established on section 3 to calculate the Analytic Torsion of $\mathbb{R}\mathrm{P}^{2m-1}$.

By Theorem 3.2 and the definition of Analytic Torsion we have:

$$
\begin{aligned}
& 2\log(T_a(\mathbb{R}\mathrm{P}^{2m-1})) = \sum_{q=0}^{2m-1}(-1)^q q\zeta'(0,\Delta^q_{\mathbb{R}\mathrm{P}^{2m-1}}) \\
= \ & \sum_{q=1}^{m-1}(-1)^q q\left[(-1)^{q-1}\log(2m-2q)+(-1)^q\log(2m-2q-2+\delta^q_{m-1})\right. \\
+ \ & \frac{4}{(2m-q-1)!(q-1)!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^q)2^{2k}\zeta_R'(-2k) \\
+ \ & \left.\frac{4}{(2m-q-2)!q!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^{q+1})2^{2k}\zeta_R'(-2k)\right] \\
+ \ & \sum_{q=m}^{2m-2}(-1)^q q\left[(-1)^{2m-2-q}\log(-2m+2+2q)+(-1)^{2m-1-q}\log(-2m+2q+\delta^q_m)\right. \\
+ \ & \frac{4}{q!(2m-q-2)!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^{2m-q-1})2^{2k}\zeta_R'(-2k) \\
+ \ & \left.\frac{4}{(q-1)!(2m-q-1)!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^{2m-q})2^{2k}\zeta_R'(-2k)\right] \\
+ \ & (-1)^{2m-1}(2m-1)\left[\log(2m-2+\delta^1_m)+(1+\delta^1_m)\log 2\right. \\
+ \ & \left.\frac{4}{(2m-2)!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^1)2^{2k}\zeta_R'(-2k)\right].
\end{aligned}
$$

By defining:

$$
\begin{aligned}
T_1 \ = \ & \sum_{q=1}^{m-1}(-1)^q q\left[(-1)^{q-1}\log(2m-2q)+(-1)^q\log(2m-2q-2+\delta^q_{m-1})\right] \\
+ \ & \sum_{q=m}^{2m-2}(-1)^q q\left[(-1)^{2m-2-q}\log(-2m+2+2q)+(-1)^{2m-1-q}\log(-2m+2q+\delta^q_m)\right] \\
+ \ & (-1)^{2m-1}(2m-1)\left[\log(2m-2+\delta^1_m)\right] \\
= \ & \sum_{q=1}^{m-1}(-1)^q q(Z_2^q+Z_2^{q+1})+\sum_{q=m}^{2m-2}(-1)^q q(Z_2^{2m-1-q}+Z_2^{2m-q})+(-1)^{2m-1}(2m-1)Z_2^1
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 \ = \ & \sum_{q=1}^{m-1}(-1)^q q\left[\frac{4}{(2m-q-1)!(q-1)!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^q)2^{2k}\zeta_R'(-2k)\right. \\
+ \ & \left.\frac{4}{(2m-q-2)!q!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^{q+1})2^{2k}\zeta_R'(-2k)\right] \\
+ \ & \sum_{q=m}^{2m-2}(-1)^q q\left[\frac{4}{q!(2m-q-2)!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^{2m-q-1})2^{2k}\zeta_R'(-2k)\right. \\
+ \ & \left.\frac{4}{(q-1)!(2m-q-1)!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^{2m-q})2^{2k}\zeta_R'(-2k)\right] \\
+ \ & (-1)^{2m-1}(2m-1)\left[\frac{4}{(2m-2)!}\sum_{k=0}^{m-1}e_{2m-2k-2}(w_1^1)2^{2k}\zeta_R'(-2k)+(1+\delta^1_m)\log 2\right],
\end{aligned}
$$

we have:

$$2\log T_a(\mathbb{R}\mathrm{P}^{2m-1}) = T_1+T_2, \tag{10}$$

where:

$$\begin{aligned} T_1 &= \sum_{q=1}^{m-1}[-\log(2m-2q)] + \sum_{q=m}^{2m-2}\left[-\log(-2m+2q+\delta_m^q)\right] \\ &+ (-1)^{2m-2}(2m-2)\log(2m-2) + (-1)^{2m-1}(2m-1)\log(2m-2) \\ &= 2\sum_{q=1}^{m-1}[-\log(2m-2q)] = 2\sum_{q=1}^{m-1}[-\log(2q)]\,. \end{aligned} \tag{11}$$

About T_2, we may use (4) substituting $\left.(2^{-2t+2k}\zeta_R(2t-2k))'\right|_{t=0}$ by x^{2k} and apply Lemma 6.6 to obtain:

$$\begin{aligned} T_2(x) &= \sum_{q=1}^{m-1}(-1)^q q\left[\tfrac{1}{2}Z_q(x) + \tfrac{1}{2}Z_{q+1}(x)\right] \\ &+ \sum_{q=m}^{2m-2}(-1)^q q\left[\tfrac{1}{2}Z_{2m-1-q}(x) + \tfrac{1}{2}Z_{2m-q}(x)\right] + (-1)^{2m-1}(2m-1)\left[\tfrac{1}{2}Z_1(x)\right] \\ &= \sum_{q=1}^{m-1}(-1)^q q\left[\tfrac{1}{2}Z_q(x) + \tfrac{1}{2}Z_{q+1}(x)\right] + (-1)^{2m-1}(2m-1)\left[\tfrac{1}{2}Z_1(x)\right] \\ &+ \sum_{q=1}^{m-1}(-1)^{2m-1-q}(2m-1-q)\left[\tfrac{1}{2}Z_q(x) + \tfrac{1}{2}Z_{q+1}(x)\right] = \\ &= \sum_{q=1}^{m-1}(-1)^q Z_q(x) + \tfrac{1}{2}(-1)^m Z_m(x) \end{aligned}$$

and since we have by (13) that

$$\begin{aligned} Z_q(x) = \frac{2}{(2m-q-1)!(q-1)!}\Bigg[&\prod_{l=1,l\neq q}^{m}(x+2m-q-l)(x+l-q) \\ &+ \prod_{l=1,l\neq q}^{m}(x-2m+q+l)(x-l+q)\Bigg], \end{aligned}$$

it follows that

$$\begin{aligned} T_2(x) &= \sum_{q=1}^{m-1}\tfrac{2(-1)^q}{(2m-q-1)!(q-1)!} \\ &\times \left(\prod_{l=1,l\neq q}^{m}(x+2m-q-l)(x+l-q) + \prod_{l=1,l\neq q}^{m}(x-2m+q+l)(x-l+q)\right) \\ &+ \tfrac{(-1)^m}{(m-1)!^2}\left(\prod_{l=1,l\neq m}^{m}(x+m-l)(x+l-m) + \prod_{l=1,l\neq m}^{m}(x-m+l)(x-l+m)\right) \end{aligned}$$

so that

$$\begin{aligned} \tfrac{(2m-2)!}{2}T_2(x) &= \sum_{q=1}^{m}(-1)^q\binom{2m-2}{q-1}\prod_{l=1,l\neq q}^{m}(x+2m-q-l)(x+l-q) \\ &+ \sum_{q=1}^{m-1}(-1)^q\binom{2m-2}{q-1}\prod_{l=1,l\neq q}^{m}(x-2m+q+l)(x-l+q). \end{aligned}$$

Defining $p = 2m - q$, we have for q varying in $\{1,\ldots,m-1\}$, that p varies in $\{m+1,\ldots,2m-1\}$ and moreover:

$$\forall l \in \mathbb{N}_m - \{q\}, \quad x - 2m + q + l = x + l - p \quad \text{and} \quad x - l + q = x + 2m - p - l,$$

so that

$$\begin{aligned}
\tfrac{(2m-2)!}{2} T_2(x) &= \sum_{q=1}^{m} (-1)^q \begin{pmatrix} 2m-2 \\ q-1 \end{pmatrix} \prod_{l=1, l\neq q}^{m} (x+2m-q-l)(x+l-q) \\
&+ \sum_{p=m+1}^{2m-1} (-1)^{2m-p} \begin{pmatrix} 2m-2 \\ 2m-p-1 \end{pmatrix} \prod_{l=1, l\neq 2m-p}^{m} (x+l-p)(x+2m-p-l)
\end{aligned}$$

Note that if $x = k$ is a natural number, $k \geq 2m$, then:

$$\forall q \in \mathbb{N}_m, \quad \prod_{l=1, l\neq q}^{m} (k+2m-q-l)(k+l-q) =$$

$$\begin{aligned}
&= \frac{(k+2m-q-1)\ldots(k+m-q)}{(k+2m-2q)} \frac{(k+m-q)\ldots(k+1-q)}{k} \\
&= \tfrac{(k+m-q)(k+2m-q-1)!}{(k-q)!} \left(\tfrac{1}{k(k+2m-2q)} \right)
\end{aligned}$$

and since for every $p \in \{m+1,\ldots,2m-1\}$, we have the following equalities:

$$\begin{pmatrix} 2m-2 \\ 2m-p-1 \end{pmatrix} = \begin{pmatrix} 2m-2 \\ p-1 \end{pmatrix},$$

$$\prod_{j=1, j\neq 2m-p}^{m} (k+j-p)(k+2m-p-j) = \frac{(k+m-p)(k+2m-p-1)!}{(k-p)!} \left(\frac{1}{k(k+2m-2p)} \right).$$

The expression, $\frac{(2m-2)!}{2} T_2(k)$ may be represented as:

$$\begin{aligned}
&\tfrac{(2m-2)!}{2} T_2(k) = \sum_{q=1}^{2m-1} (-1)^q \begin{pmatrix} 2m-2 \\ q-1 \end{pmatrix} \tfrac{(k+m-q)(k+2m-q-1)!}{(k-q)!} \tfrac{1}{k(k+2m-2q)} \\
&\Rightarrow T_2(k) = 2 \sum_{q=1}^{2m-1} (-1)^q \begin{pmatrix} 2m-2 \\ q-1 \end{pmatrix} \tfrac{(k+2m-q-1)!}{(k-q)!(2m-1)!} (2m-1) \tfrac{k+m-q}{k(k+2m-2q)} \\
&\Rightarrow T_2(k) = 2m-1 \sum_{q=1}^{2m-1} (-1)^q \begin{pmatrix} 2m-2 \\ q-1 \end{pmatrix} \begin{pmatrix} k+2m-q-1 \\ 2m-1 \end{pmatrix} \tfrac{2k+2m-2q}{k(k+2m-2q)} \\
&\Rightarrow T_2(k) = 2m-1 \sum_{q=1}^{2m-1} (-1)^q \begin{pmatrix} 2m-2 \\ q-1 \end{pmatrix} \begin{pmatrix} k+2m-q-1 \\ 2m-1 \end{pmatrix} \left(\tfrac{1}{k} + \tfrac{1}{k+2m-2q} \right) \\
&\Rightarrow T_2(k) = -(2m-1) \sum_{q=0}^{2m-2} (-1)^q \begin{pmatrix} 2m-2 \\ q \end{pmatrix} \begin{pmatrix} k+2m-q-2 \\ 2m-1 \end{pmatrix} \left(\tfrac{1}{k} + \tfrac{1}{k+2m-2q-2} \right)
\end{aligned}$$

and substituting q by $2m - q - 2$, we obtain:

$$T_2(k) = -(2m-1) \sum_{q=0}^{2m-2} (-1)^q \begin{pmatrix} 2m-2 \\ q \end{pmatrix} \begin{pmatrix} k+q \\ 2m-1 \end{pmatrix} \left(\frac{1}{k} + \frac{1}{k+2q-(2m-2)} \right).$$

Now, applying Lemma 6.3, we obtain:

$$T_2(k) = -2m.$$

So $T_2(x)$ is a polynomial satisfying $T_2(k) = -2m$ for all $k \in \mathbb{N} - \mathbb{N}_{2m-1}$ and we may conclude $T_2(x)$ is the constant polynomial $T_2(x) = -2m$.

Therefore we have:

$$T_2 = -2m(2^{-2t}\zeta_R(2t))'|_{t=0} = -2m(\log 2) + 2m\log(2\pi). \tag{12}$$

By using the expressions (10), (11) and (12), we obtain:

$$T_a(\mathbb{R}\mathrm{P}^{2m-1}) \quad = \quad \exp\left(\tfrac{1}{2}(T_1+T_2)\right) = \frac{1}{\prod\limits_{q=1}^{m-1} 2q}\frac{(2\pi)^m}{2^m} = \frac{\pi^m}{2^{m-1}(m-1)!}.$$

This proves the following Theorem, where we use the volume of $\mathbb{R}\mathrm{P}^{2m-1}$, see [1]:

Theorem 5.1. *The Analytic Torsion of odd dimensional real projective spaces is given by:*

$$T_a(\mathbb{R}\mathrm{P}^{2m-1}) = \frac{\pi^m}{2^{m-1}(m-1)!} = \frac{\mathrm{Vol}(\mathbb{R}\mathrm{P}^{2m-1})}{2^{m-1}}.$$

The Analytic Torsion of $\mathbb{R}\mathrm{P}^{2m-1}$ in relation to a unitary representation of $\pi_1(\mathbb{R}\mathrm{P}^{2m-1})$ is obtained in terms of the Analytic Torsion of $\mathbb{R}\mathrm{P}^{2m-1}$, calculated above, and the Analytic Torsion of S^{2m-1}, calculated in [15, Theorem 4.2], by a similar reasoning to the one applied on Proposition 3.3. The result is described below:

Theorem 5.2. *The Analytic Torsion of $\mathbb{R}\mathrm{P}^{2m-1}$ in relation to a representation*

$$\rho \simeq \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{\gamma} \oplus \underbrace{\sigma_2 \oplus \cdots \oplus \sigma_2}_{N-\gamma},$$

is given by:

$$T_a(\mathbb{R}\mathrm{P}^{2m-1}, \rho) = T_a(\mathbb{R}\mathrm{P}^{2m-1})^{2\gamma-N} \cdot T_a(S^{2m-1})^{N-\gamma}.$$

6 Appendix

We present in this appendix some results from other references that are necessary for the study developed in this text. The first of these results was proved on [2, Theorem 5.3] and on [14, Theorem 1.2(ii)] and establishes an identity involving Bernoulli polynomials $B_n(x)$.

Lemma 6.1. *For $p,q \in \mathbb{Z} \cap [0,+\infty)$ and $x+y+z=1$, we have:*

$$\begin{aligned} &(-1)^p \sum_{j=0}^{p} \binom{p}{j} x^{p-j} \frac{B_{q+1+j}(y)}{q+1+j} + (-1)^q \sum_{j=0}^{q} \binom{q}{j} x^{q-j} \frac{B_{p+1+j}(z)}{p+1+j} \\ = \; & \frac{(-x)^{p+q+1} p!q!}{(p+q+1)!}. \end{aligned}$$

The second result is a combination of Theorem C and part of Theorem B of [9] and establishes properties of the zeta function associated to two polynomials $P(x)$ and $Q(x)$, which is defined for $\mathrm{Re}(s)$ sufficiently large by:

$$\zeta(s;P,Q) := \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)^s}.$$

Lemma 6.2. *Given two polynomials $P(x)$ and $Q(x)$ with complex coefficients such that every root of $Q(x)$ is in $\mathbb{C} - \{x \in \mathbb{R} \mid x \geqslant 1\}$, the associated zeta function, $\zeta(s;P,Q)$ admits meromorphic extension to $\mathbb{C}$ and this extension is holomorphic at non-positive integers.*

The third result was proved by Weng and You on [15, Lemma 3].

Lemma 6.3. *Let $k,m \in \mathbb{N}$. It is valid that:*

$$\sum_{p=0}^{m} (-1)^p \left(\begin{array}{c} m \\ p \end{array} \right) \left(\begin{array}{c} k+p \\ m+1 \end{array} \right) = (-1)^m k,$$

and

$$\sum_{p=0}^{m} (-1)^p \left(\begin{array}{c} m \\ p \end{array} \right) \left(\begin{array}{c} k+p \\ m+1 \end{array} \right) \frac{m+1}{k+2p-m} = \frac{1+(-1)^m}{2}$$

The next three lemmas were proved by the author on [12] and are fundamental for proving the determinant formula of section 3. Lemma 6.4 corresponds to [12, Lemma 3.2], Lemma 6.5 corresponds to [12, Lemma 4.1], and Lemma 6.6 combines the statement of [12, Lemma 4.2] and an expression obtained in the proof of this lemma.

Lemma 6.4. *Let $n \in \mathrm{N}$ and $m = [\![\frac{n+1}{2}]\!]$. For $q \in \mathbb{N}_m$ and $S = \{(l-q)(n+1-q-l) \mid l \in \mathbb{N}_m - \{q\}\}$ we have $S = S_1 \cup S_2$, where:*

$$S_1 = \{-k(k+n+1-2q) \mid 1 \leqslant k \leqslant q-1\} \quad \textit{and}$$

$$S_2 = \{-k(k-n-1+2q) \mid 1 \leqslant k \leqslant n-2q\}.$$

Lemma 6.5. *For every $q \in \mathbb{N}_m$ and $k \in \mathbb{N} \cup \{0\}$, defining $u_l^q = (l-q)(2m-q-l)$ and $u^q = (u_l^q)_{l \in \mathbb{N}_m - \{q\}}$, we have:*

$$\xi_{q,k}^{o}(s) = \sum_{j=0}^{m-1} \frac{2e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!} \frac{1}{(k+q)^{s-j}(k+2m-q)^{s-j}},$$

where $e_{m-j-1}(u^q)$ are elementary symmetric polynomials on u^q, i. e.

$$\prod_{l=1, l\neq q}^{m} (x+u_l^q) = \sum_{j=0}^{m-1} e_{m-j-1}(u^q) x^j,$$

and $\xi_{q,k}^{o}(s)$ are defined by expression (2) on section 3 of this text.

Lemma 6.6. *Let*

$$Z_q(x) = 4\sum_{j=0}^{m-1} \tfrac{e_{m-j-1}(u^q)}{(2m-q-1)!(q-1)!} \sum_l \left(\begin{array}{c} j \\ 2l \end{array} \right) (2m-2q)^{2l} x^{2j-2l},$$

with $u^q = (u_l^q)_{l\in\mathbb{N}_m-\{q\}}$ *and* $u_l^q = (l-q)(2m-q-l)$. *Then:*

$$\begin{aligned} Z_q(x) &= \frac{2}{(2m-q-1)!(q-1)!}\left[\prod_{l=1,l\neq q}^{m}(x+2m-q-l)(x+l-q)\right. \\ &\quad \left. + \prod_{l=1,l\neq q}^{m}(x-2m+q+l)(x-l+q)\right] \qquad (13) \\ &= \frac{4}{(2m-q-1)!(q-1)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_1^q)x^{2k} \end{aligned}$$

where

$$\begin{aligned} w_1^q &= (l-q)_{l\in\mathbb{N}_m-\{q\}} \times (2m-q-l)_{l\in\mathbb{N}_m-\{q\}} \\ &= (1-q,\ldots,\widehat{q-q},\ldots,m-q,2m-q-1,\ldots,\widehat{2m-2q},\ldots,2m-q-m). \end{aligned}$$

The final lemma was proved by the author on [11, Corollary 2.1] and is used in this text exclusively on Propositions 3.3 and 5.2 to deal with the case $m = 1$. For this lemma we consider an arbitrary n-fold Riemannian covering $\pi : \tilde{M} \to M$ and decompose the space $E(\lambda,\Delta^p_{\tilde{M}})$ in terms of $\pi^*E(\lambda,\Delta^p_{M,\sigma_i})$, where σ_i, $i \in \mathbb{N}_r$ are representatives of the irreducible representations of the group of deck transformations of the covering. On this lemma we use use $P_{i,j}$ to denote the projections of differential forms onto their coordinates:

$$P_{i,j} : C^\infty(\Lambda^p\tilde{M}\otimes\mathbb{C}^{k_i}) \to C^\infty(\Lambda^p\tilde{M}); \quad \eta\otimes(f_1,\ldots,f_i) \mapsto f_j\cdot\eta.$$

Lemma 6.7. *Let* $\pi : \tilde{M} \to M$ *be an n-fold Riemannian covering. For every* $\lambda \in \mathbb{R}$ *the space* $E(\lambda,\Delta^p_{\tilde{M}})$ *may be decomposed in the form:*

$$E(\lambda,\Delta^p_{\tilde{M}}) = \bigoplus_{i\in\{1,\ldots,r\},\, j\in\{1,\ldots,k_i\}} P_{i,j}(\pi^*E(\lambda,\Delta^p_{M,\sigma_i})).$$

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It is important to mention that this project was developed alongside the reference [6]. These authors presented me a preliminar/unfinished version of [6] and asked me to finish it with the level of generality present in this manuscript and [13], but I did not understand their reasoning by the time, so I decided to restart from zero using a different approach. During the time this research was developed, professor Hartmann

was on Germany, professor Spreafico was on Italy and I was on Brazil, so I was working independently. When professor Hartmann finished his studies on Germany and came back to Brazil, he decided to finish and submit [6] with M. Spreafico and allowed me to submit my text alone. I did not express at any moment during the course that I wanted to publish my thesis alone, but I stayed very happy with his decision, even though I recognize they helped me, so I was not really working alone. In retribution to their help I tried to help them on [6] as much as I could.

Conflicts of interest

Conflicts of Interest: None

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