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José Jansen

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Purposing an Algebraic Solution to the Four-Color Problem

José Ulisses Jansen*[1]

[1]Institute of Chemistry, University of Campinas, 13083-970 Campinas, SP, Brazil.

https://orcid.org/0000-0002-7090-2224

***ujansen@unicamp.br**

Abstract

The Four-Color Theorem was originated with the coloring of Countries in a MAP and it was a challenging problem that remained open since 1853 for more than 170 years. By the end of Sec XX, this problem was solved using computational tools but until today there is no algebraic proof of it. In this article, the original problem of coloring MAPS over a Spherical Surface is briefly reviewed. A Spherical MAP is converted into a Planar MAP using polar coordinates and the frontiers of the Countries are described as real implicit equations and then deployed from the real space into the complex space. In the complex space the rules to color MAPs are described as system of algebraic equations and inequations. One example of MAP is solved (colored) and the explanation about why these systems are solvable is done. Beginning from the example, a general theory to coloring MAPs is derived. As all the transformations used admits inverse, the obtained planar MAP solution can be reversed as a solution to the Spherical MAP. All operations involve simple algebraic transformations and some Calculus concepts.

Key-words: Four-Color Theorem; Complex deployment.

1. Introduction

The Four-Color Theorem was proposed in 1853 by Francis Guthrie and received attention by great mathematicians for many years and remained unsolved (1), (2), (3) until 1976 when a disruptive solution was purposed (4). This solution was firstly refused and the employed methodology suffered revisions ex (5), (6) until the work of Gonthier (7) was finally accepted. The use of computational tools as a central part of the proof instead of the use of pure abstract arguments causes discomfort to the purists even nowadays. In the present text, it is used the fact that a constant radius Spherical MAP has a biunivocal relation with the (θ, φ) spherical coordinates. In a (θ, φ) plane, the frontiers of each Country may be described by equations. A selection rule-based in clockwise orientation sense is used to choose the positive functions set and based on this selection a complex space is generated, converting each positive real equation into four complex equations. After this preparation, contour conditions overall connected frontier segments are modeled by a system of equations and inequations. This system is demonstrated to be cyclic and solvable in one color (the trivial problem, one Country, and one-color solution that is not treated) until four colors (the most complicated case, which is treated here).

2. Development of the Theory

2.1 – The representation of a MAP over a Spherical Surface and its Polar Coordinates representation.

As known (8, p 123) Polar coordinates transformations ($x = sin\theta cos\varphi; y = sin\theta sin\varphi; z = cos\theta$) establish a biunivocal correspondence between the points belonging to a unitary Sphere and their coordinates in the (θ, φ) plane.

2.1.1 Defining a Planar MAP using Implicit functions to define frontiers.

Especially for two-variable functions, the Implicit Function Theorem (FIT) (9, p. 480-485) guarantees that under mild conditions $f(\theta, \varphi) = 0$ may be reduced to $\theta = r(\varphi)$ or $\varphi = r(\theta)$. Then, the domain (θ, φ) can be decomposed into different Countries using implicit functions to define their frontiers as in Fig 1.

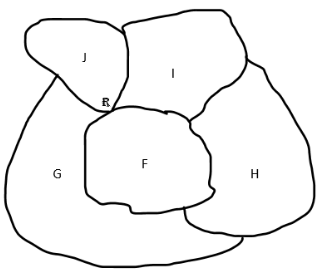

Fig 1: Countries F, G, H, I and J separated each other by their correspondent Implicit functions $f(\theta, \varphi) = 0;\ g(\theta, \varphi) = 0; h(\theta, \varphi) = 0;\ i(\theta, \varphi) = 0;\ j(\theta, \varphi) = 0$ and the null segment measure point ℝ

As known from Calculus, to each closed frontier there are always two implicit equations $f(\theta, \varphi) = 0$ and $-f(\theta, \varphi) = 0$ associated with the frontiers. To avoid that two different functions delimit a unique region, a positive sign is attributed (here) to the function that has a positive sign at the right side of a clockwise dislocation over the frontier of one Country. These positive signed functions are named $f^p{}'s$. Every f^p function is unique and can be deployed into four different complex functions ($f^p + i\,f^p$, $f^p - i\,f^p$, $-f^p + i\,f^p$, $-f^p - i\,f^p$). The four equations ($f^p + i\,f^p = f^p - i\,f^p = -f^p + i\,f^p = -f^p - i\,f^p =$

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0) have de same trace in the (θ, φ) surface but they have different signs at their right and left sides. The functions ($f^p + i\, f^p$, $f^p - i\, f^p$, $-f^p + i\, f^p$, $-f^p - i\, f^p$) will be denoted as (f^{++} , f^{+-} , f^{-+} , f^{--}).

2.1.2 Some definitions and Sub-theorems

Some definitions and Sub-theorems are important to the subsequent derivations:

2.1.2.1-) Each real equation ($f(\theta, \varphi) = 0$) has only one internal color (a direct consequence of the two variable implicit real continuous function and the selection of f^p done in 2.1.1).

2.1.2.2-) The four complex equations $\{ f(\theta,\varphi) + if(\theta,\varphi) = 0\,;\ +f(\theta,\varphi) - if(\theta,\varphi) = 0;\ -f(\theta,\varphi) + if(\theta,\varphi) = 0;\ -f(\theta,\varphi) - if(\theta,\varphi) = 0 \}$ have the same trace in the surface (θ, φ) (by definition), then every frontier segment may be represented by four complex equations.

2.1.2.2.a) To every segment of a frontier, two of the four complex equations are assigned, one associated with the internal (left) side and the other associated with the external (right) side of the segment (by definition).

2.1.2.2.b) Adjacent segments on the external side of any Country represent different Countries then they are represented by different complex equations.

2.1.2.3 -) A continuous, connected (closed), non-self-crossing line (frontier) can be subdivided only in an even or in an odd number of segments (by construction).

2.1.2.3.a -) If a continuous, connected (closed), non-self-crossing line is divided into two or more **even** number of segments, **it can ever be colored two colors** (by inspection).

2.1.2.3.b -) If a continuous, connected, non-self-crossing line is divided into three or more **odd** number of segments, it **can ever be colored with three colors** (by inspection).

2.1.3-) An example of a 4-color 5-Countries MAP coloring using the complex space function and the respective enumeration rules applied to the frontier's segments.

Algorithm:

a.) In a selected MAP, put all countries uncolored.

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b.) Enumerate (nominate) all countries. Enumerate all frontier segments (even in null measure segments as point $\mathbb{R}$ in Fig 1) of all countries according to their Country's boundaries.

c.) Select a Country M

d.) Beginning with M, write the associated equation system (for the internal colors) and the inequation system (for the external colors) for M and all Countries neighbors of M. With this procedure, expand subsequently the equation and inequation system for all Countries of the MAP.

e.) Adopt a color for one Country and solve all the equations and inequations system.

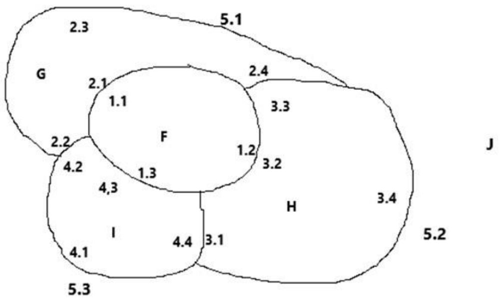

Fig 2: Example of frontier's segments indexation of a 5 (F, G, H, I, and J MAP).

Equations	Inequations
$1-)\ f^{1.1} = f^{1.2} = f^{1.3}$	$5-)\ g^{2.1} \neq h^{3.2}$; $6-)\ h^{3.2} \neq i^{4.3}$; $7-)\ i^{4.3} \neq g^{2.1}$
$2-)\ g^{2.1} = g^{2.2} = g^{2.3} = g^{2.4}$	$8-)\ g^{2.4} \neq h^{3.3}$; $9-)\ h^{3.1} \neq i^{4.4}$; $10-)\ i^{4.2} \neq g^{2.2}$
$3-)\ h^{3.1} = h^{3.2} = h^{3.3} = h^{3.4}$	$11-) f^{1.1} \neq g^{2.1}$; $12-)\ f^{1.2} \neq h^{3.2}$; $13-)\ f^{1.3} \neq i^{4.3}$
$4-)\ i^{4.1} = i^{4.2} = i^{4.3} = i^{4.4}$	$14-)\ j^{5.1} \neq g^{2.3}$; $15-)\ j^{5.2} \neq h^{3.4}$; $16-)\ j^{5.3} \neq i^{4.1}$

At the beginning, all equations and inequations have 4 degrees of freedom. In the system above, suppose that 1-) $f^{1.1} \in f^{++}$ then 11-) $\Rightarrow$ $g^{2.1} \in Complement(f^{++}) = \{f^{+-}, f^{-+}, f^{--}\}$. Then suppose that $g^{2.1} \in f^{+-}$ then 5-), 12-) and 1-) $\Rightarrow$ $h^{3.2} \in Complement(f^{++}, f^{+-}) = \{f^{-+}, f^{--}\}$. Then suppose that $h^{3.2} \in f^{-+}$ then by 6-), 7-) 13-) and 1-) $\Rightarrow$ $i^{4.3} \in Complement(f^{++}, f^{+-}, f^{-+}) = \{f^{--}\}$.

With ($f^{1.1}$, $g^{2.1}$, $h^{3.2}, i^{4.3}$) determined, all remaining equations and inequations are reduced to one degree of freedom and the coloring of the MAP's is solved. It is important to note that the solution to any external contour of $\{g^{2.3};\ h^{3.4};\ i^{4.1}\}$ will be realized in one of the $\{g^{2.3};\ h^{3.4};\ f^{1.1}\}$ or $\{f^{1.1};\ h^{3.4};\ i^{4.1}\}$ or $\{g^{2.3};\ f^{1.1};\ i^{4.1}\}$ subsets. In the case of Fig 2 $j^{5.1}$, $j^{5.2}$ and $j^{5.3} \in f^{++}$.

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Revising the example of Fig 2 is enough to solve the Reduced System of Equations and Inequations as presented in Fig 3.

Equations (internal restraints)	Inequations (external restraints)	(internal X external restraints)
$1-)\ \ f^{1.1} = f^{1.2}$	$5-)\ g^{2.1} \neq h^{3.2};$	$11-)\ f^{1.1} \neq g^{2.1}$
$1'-)\ f^{1.2} = f^{1.3}$	$6-)\ h^{3.2} \neq i^{4.3};$	$12-)\ f^{1.2} \neq h^{3.2}$
$1''-)\ f^{1.3} = f^{1.1}$	$7-)\ i^{4.3} \neq g^{2.1};$	$13-)\ f^{1.3} \neq i^{4.3}$

Fig 3: Reduced Equation and Inequation System.

An initial condition to solve the system is required f. ex. the internal color $f^{1.1}$. Then as the inequation system 5-), 6-) and 7-) constitute an odd cycle, because of 2.1.2.3-b) it requires 3 colors to be solved. After this, all other inequations 8-) to 16-) have 1 degree of freedom. In this case there are 4*3*2*1 = 24 equivalent color configurations. For a general problem, there are additional colors configurations due to the extra degrees of freedom due to the presence of even cycled Countries.

In synthesis, the "machinery" works, in part, because the **external restraints of each Country** are always reducible to even and/or odd cycles then they can always be solved in two or three colors.

2.1.4 Generalizing the theory – from the particular to the general

From example 2.1.3-) a general theory can be deduced observing that:

2.1.5-a) For an n Country MAP, the internal $f^{i.*}$ colors of the segments of each Country i are governed by one **equation** set of order k $\Rightarrow$ $f^{i.1} = f^{i.2} = \ldots f^{i.k}$ where $i = 1\ to\ n$ where k is dependent on i ($k = k(i)$).

2.1.5-b) For each $f^{i.k(i)}$ internal representation there is one $f^{j.k(j)}$ external representation. The $k(i)$ order of the $i - esime$ Country is the same order of the **inequation I** set { $g^{2.1} \neq h^{3.2}; .. \neq o^{j.k(j)}$ } that governs the external cycle ($j = n$) of the $i - esime$ Country.

Then, any MAP is described by a set of Equations and Inequations divided into distinct subsets as in Fig 4.

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Equations	Inequations 1	Inequations 2
$f^{1.1} = f^{1.2}$	$g^{2.1} \neq h^{3.2};$	$f^{1.1} \neq g^{2.1}$
$f^{1.2} = f^{1.3}$	$h^{3.2} \neq i^{4.3};$	$f^{1.2} \neq h^{3.2}$
$f^{1.3} = f^{1.1}$	$i^{4.3} \neq h^{2.1};$	$f^{1.3} \neq i^{4.3}$
$g^{2.1} = g^{2.2}$	$f^{1.1} \neq i^{4.2};$	$g^{2.1} \neq f^{1.1}$
$g^{2.2} = g^{2.3}$	$i^{4.2} \neq j^{5.1};$	$g^{2.2} \neq i^{4.2}$
$g^{2.3} = g^{2.4}$	$j^{5.1} \neq h^{3.3};$	$g^{2.3} \neq j^{5.1}$
...	...;	... ;
$g^{2.k} = g^{2.1}$	$o^{2.1} \neq f^{1.1};$	$g^{2.k} \neq o^{2.1}$
...	...;	... ;
...	...;	... ;
$p^{k.1} = p^{k.2}$	$q^{l.1} \neq r^{m.2};$	$p^{k.1} \neq q^{l.1}$
.... ;	.... ;	....
$p^{k.j} = p^{k.1}$	$t^{n.j} \neq q^{l.1};$	$p^{k.j} \neq t^{n.j}$

Fig 4: View of a generic MAP described in an Equation and Inequation System

For an n Country MAP there are even and odd k order Countries. The subset of Equations determines that one color define the (internal) color of each $i-esime$ Country. The subset of Inequations 2 determines that the colors of the neighboring Countries are complementary to the (internal) color of the $i-esime$ Country color. Then the use of 2.1.5-b) implies that all Inequalities 1 composed by odd order cycles are solved with 3 colors. Then, the resolution of the MAP system needs one color for each internal color (Equation Subset) and 3 complementary colors for the external colors (odd Inequation 1 subset).

Finally, it is necessary to prove that the color set that colors one subset is the same set that colors others all subsets. For this, suppose that f^* exist as a fifth color. Then f^* can not belong to $\{f+i*f;\ -f+i*f;\ f-i*f;\ -f-i*f\}$ set **then f^* is not a real function (because every real function can be deployed into four complex functions) and this conclusion contradicts the Implicit Function Theorem** that guarantees the existence of implicit real f^* functions in $\mathcal{R}^2$ ($\mathcal{R}^n$). Then, in this Space, f^* exist as a real function and it belongs to $\{f+i*f;\ -f+i*f;\ f-i*f;\ -f-i*f\}$ set and a fifth color does not exist.

3. Conclusion:

The Four-Color Theorem has an algebraic demonstration formulating it as an Equation and Inequation System. As part of the resolution, the deployment of Real Equations into Complex Equations is presented as an innovative view of Real Equations. The technique for doing this transformation and the rules for segments enumeration and the solving method for the complex Equation and Inequation System is presented as an example for

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this modeling. The consequences of the observations 2.1.2.3.b -), 2.1.5-a) and 2.1.5-b) are determinant to the solution. The Implicit Function Theorem was used due to the simplicity of the demonstration it gives. A different approach is to divide each segment into new real functions or approximate them by piecewise linear functions, splines, besier functions, etc and reconstruct each segment and each frontier as the union of these functions. With this reconstruction in hand, it is possible to apply the complex deployment for the entire set of equations and inequations. The above derivation seems too cumbersome and is open as a challenge to the reader. The author believes that the deployment of Real equations into Complex equations can be used in theoretical and/or applied fields of Mathematics and Physics. As a non-mathematician, the author recognizes that the text is too resumed and may require more formalism. There are many open opportunities to be explored, even in correcting eventual mistakes did by the author.

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