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Determinants and Cheeger-Müller Theorem on even dimensional real projective spaces*

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Abstract

We establish formulae for the regularized determinant of the Laplacian and twisted Laplacian on forms over even dimensional real projective spaces and use them to explicitly calculate the analytic torsion of these spaces with orthogonal and unitary representations of their fundamental group, obtaining the first example to show that this torsion is non-trivial for cyclic complexes associated to even dimensional non-orientable manifolds. Moreover, as discussed in the introduction, the several proofs of the Cheeger-Müller Theorem given throughout years left one specific question unanswered and we answer this question for this type of space.
MSC2010: 58J52, 57Q10, 11M36. **Keywords** Cheeger-Müller Theorem; Regularized Determinants; Real Projective Spaces

1 Introduction

Let M be a closed Riemannian manifold and let $\rho : \pi_1(M) \to GL(E)$ be a representation of the fundamental group of M, where E is a finite dimensional real or complex vector space. The spectral zeta function of the twisted Laplacian $\Delta^q_{M,\rho}$ is defined in terms of the spectrum of this operator by extending meromorphically the function defined for $\mathrm{Re}(s)$ sufficiently large by the following formula:

$$\zeta(s, \Delta^q_{M,\rho}) = \sum_{\lambda>0} \dim(E(\lambda, \Delta^q_{M,\rho}))\lambda^{-s}, \quad \mathrm{Re}(s) \gg 0,$$

where $E(\lambda, \Delta^q_{M,\rho})$ denotes the eigenspace associated to the eigenvalue λ.

This function was created by Minakshisundaram and Pleijel on [13] and has several applications, in particular, its derivative at zero provides us with the important concept of regularized determinant of the Laplacian, $\det(\Delta^q_{M,\rho})$, according to the following expression:

$$\det(\Delta^q_{M,\rho}) := \exp(-\zeta'(0, \Delta^q_{M,\rho})).$$

The regularized determinant by itself has several applications, both in Physics and Mathematics. Some applications of this concept to Physics are discussed

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on [6], where the authors establish a formula for determinants on a specific class of differential operators, compare it to the already known result for the determinant on functions over S^2 (on the last paragraph of subsection 4.5) and mention some possible applications of their work (on section 7). About applications in Mathematics, on [21] the author uses determinants of the Laplacian on spheres to generalize a formula about multiple gamma functions; and on [19] we have one of the most important applications of regularized determinants to mathematics, since in this text the authors use the determinant of the twisted Laplacian to define the analytic torsion $T_a(M;\rho)$ of a closed Riemannian manifold M in relation to an orthogonal representation ρ of its fundamental group. The precise definition of $T_a(M;\rho)$ is given by:

$$T_a(M,\rho) = \exp\left(\frac{1}{2}\sum_{q=0}^{n}(-1)^q q\zeta'(0,\Delta^q_{M,\rho})\right).$$

The definition of analytic torsion was created by Ray and Singer [19] as an analytic version for Reidemeister torsion (see section 5 for a discussion about this concept), and these authors conjectured that for odd dimensional orientable manifolds with orthogonal representation of the fundamental group these torsions were equivalent, what had already been proved for lens spaces on [18] and for even dimensional orientable manifolds as a consequence of Theorem 2.3 of [19].

It is important to mention that these authors followed Milnor's definition of Reidemeister torsion, given on section 8 of [12], but they extended this definition, since Milnor considered exclusively acyclic complexes, i. e., complexes whose homology is trivial, and Ray and Singer considered also the cyclic cases. It is also important to mention that both [12] and [19] considered homology when dealing with Reidemeister torsion and this was specially relevant on [19], since these authors needed Poincaré duality to define the bases used for homology groups in the cyclic case.

On 1978, Werner Müller [14] published a proof for the conjecture of Ray-Singer, which is known as the Cheeger-Müller Theorem, and he considered exclusively orientable manifolds throughout the text. Moreover, he made a slight change in the problem proposed by Ray and Singer, since he considered the Reidemeister torsion of the cohomology complex associated to a manifold rather than homology. On 1979, Cheeger [5] published an independent proof of the same problem and also considered cohomology when proving the theorem, even though his proof does not require orientability.

The Cheeger-Müller Theorem was studied by Bismut and Zhang on 1992, [1], who also considered cohomology when dealing with Reidemeister torsion, following [5] and [14], but these authors extended this result to odd dimensional closed Riemannian manifolds with unimodular representation of the fundamental group, and constructed an integral formula which describes the relation between the two torsions on the even dimensional case.

On 1993, [15], Müller returned to the conjecture proposed by Ray and Singer, this time using homology to study Reidemeister torsion, and he proved that both torsions have the same value on odd dimensional manifolds with unimodular representation of the fundamental group, what does not happen in general in the even dimensional case, as examplified on page 732 of the same text, [15]. However, the approach used by Müller when dealing with cyclic complexes required

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the use of orientability because to describe the volumes used in the cyclic case, first paragraph of page 729, the author needed Poincaré duality and there is no indication of how to replace this concept in the non-orientable setting. This specific choice of volumes is fundamental for Theorem 2.15 and Corollary 2.16 of [15], which were originally proved by Ray and Singer on [19] for orthogonal representations and were extended by Müller [15] to the unimodular case. In section 3, the core part of the proof of Cheeger-Müller Theorem, the specific choice of volumes also plays an important role, specially on page 745.

The Cheeger-Müller Theorem was studied again on [3], [4] and [10], but these authors also considered cohomology when dealing with Reidemeister torsion.

In this text we prove that Cheeger-Müller Theorem is valid for even dimensional real projective spaces in the sense discussed on [15] by explicitly calculating their analytic and Reidemeister torsions, see Theorem 5.1. To calculate the analytic torsion we first establish formulae for the regularized determinant of the Laplacian, Theorem 3.1 and Corollary 3.3, and we use these formulae to obtain the desired torsion, Theorem 4.2. This reasoning was inspired by [22], even though these authors did not establish regularized determinant formulae. To calculate the Reidemeister torsion we first establish homology bases for odd dimensional projective spaces, which are orientable, and we use these bases as references for our specific choice in the even dimensional case. It is important to mention that these torsions are not trivial for $\mathbb{RP}^{2m}$, contrary to what happens for orientable even dimensional manifolds, as expressed by Proposition 2.9 of [15]. We have no knowledge of any prior reference showing an example of this fact for even dimensional non-orientable manifolds with cyclic representation of the fundamental group.

Our study is divided in the following way: section 2 is dedicated to establishing the notation that will be used throughout the text as well as some fundamental preliminary results for our determinant formulae, the most important ones are the decomposition of the spaces of forms into $SO(n+1)$-modules, Table 2, and the derivative at zero of a specific Dirichlet series, Lemma 2.1. Section 3 is devoted to establishing the formulae for regularized determinants, which will be used on section 4 to calculate the analytic torsion of $\mathbb{RP}^{2m}$. On section 5, the final section, we define the Reidemeister torsion of real chain complexes and describe how to fix bases for the homology groups of $\mathbb{RP}^{2m-1}$, following Müller's approach to the problem, [15]. We then use these data to determine the specific bases that would be used to calculate the Reidemeister torsion of odd dimensional projective spaces and adapt these bases to the even dimensional case to obtain the Reidemeister torsion of $\mathbb{RP}^{2m}$ in relation to this particular choice of bases. Since the results obtained on sections 4 and 5 are equal, we conclude the validity of Cheeger-Müller Theorem for $\mathbb{RP}^{2m}$ in the setting discussed in this text.

2 Prerequisites for the formula

The first step for establishing our formula for regularized determinants is to determine the spectrum and multiplicity of eingenvalues of the Laplacian on forms over even dimensional real projective spaces. To achieve this objective we outline part of the study of Ikeda and Taniguchi [9].

In section 6 of [9] the authors define the spaces of harmonic polynomial forms

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on $\mathbb{R}^{n+1}$ as the sets H_k^q of $\alpha \in C^\infty(\mathbb{R}^{n+1})$ such that, for $\overline{\Delta}$ the Laplace operator on forms over $\mathbb{R}^{n+1}$ and δ_0 the formal adjoint of the exterior differential, α satisfies $\overline{\Delta}\alpha = 0$, $\delta_0\alpha = 0$ and $\alpha = \sum_{0\leqslant i_1<\cdots<i_q\leqslant n+1} \alpha_{i_1,\ldots,i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}$, where $q \in \{0,\ldots,n\}$ ($q = 0$ corresponds to functions), $k \in \mathbb{N} \cup \{0\}$ and $\alpha_{i_1,\ldots,i_q}$ are homogeneous polynomials of degree k.

Moreover, they prove in Corolary 6.6 that for $i : S^n \to \mathbb{R}^{n+1}$ the inclusion map, we have for every $q \in \{0,\ldots,n\}$ that $i^* : \cup_{k\geq 0} H_k^q \to C^\infty(\Lambda^q(S^n))$ is injective and has its image dense in $C^\infty(\Lambda^q(S^n))$.

Considering the natural actions A_1 and A_2 of $SO(n+1)$ on $C^\infty(\Lambda^q(\mathbb{R}^{n+1}))$ and on $C^\infty(\Lambda^q(S^n))$, respectively, we see that H_k^q and $i^*H_k^q$ are $SO(n+1)$-modules, and moreover, $i^*A_1 = A_2 i^*$. So, when we decompose H_k^q in $SO(n+1)$-irreducible modules we are also decomposing $i^*H_k^q$.

Ikeda and Taniguchi [9] provide us with this decomposition. The resulting data is summarized on Table 1, that is also present on [17], by using the maximal weights corresponding to the root system $\prod = \{\pm e_i \pm e_j \mid i \neq j\}$, the positive roots $\prod_+ = \{e_i - e_j, e_i + e_j \mid i < j\}$ and simple roots $\{e_1 - e_2, e_2 - e_3, \ldots, e_{m-1} - e_m, e_{m-1} + e_m\}$. On this table we use the notation $E(k\Lambda_1 + \Lambda_q)$ to represent the $SO(n+1)$-module corresponding to the weight $k\Lambda_1 + \Lambda_q$, with the convention that for $1 \leqslant q \leqslant m$, $E(-1\Lambda_1 + \Lambda_q) = E(-1\Lambda_1 + \Lambda_m^-) = E(-1\Lambda_1 + \Lambda_m^+) = \{0\}$ is the trivial module. Note that we follow the roots used by Weng and You on section 2 of [22], which contrast with the corresponding roots adopted on [9].

Table 1: Decomposition of the space of forms into $SO(n+1)$-modules on even dimensional spheres

$n = 2m$	
q	H_k^q
$q = 0$	$E(k\Lambda_1)$
$1 \leqslant q \leqslant m-1$	$E(k\Lambda_1 + \Lambda_q) \oplus E((k-1)\Lambda_1 + \Lambda_{q+1})$
$q = m$	$E(k\Lambda_1 + 2\Lambda_m) \oplus E((k-1)\Lambda_1 + 2\Lambda_m)$
$m+1 \leqslant q \leqslant 2m-1$	$E((k-1)\Lambda_1 + \Lambda_{n-q}) \oplus E(k\Lambda_1 + \Lambda_{n-q+1})$
$q = 2m$	$E((k+1)\Lambda_1)$

Besides using the results from [9], we also use the theory of [7, section 3.9], which establishes that if M is a Riemannian manifold whose fundamental group is cyclic of order l and $(\rho_s)_{0\leqslant s\leqslant l}$ are representatives for the irreducible representations of $\pi_1(M)$, then

$$E(\lambda, \Delta^q_{\tilde{M}}) = \oplus_s E_{eq}(\lambda, \Delta^q_{M,\rho_s}) \tag{1}$$

where $\tilde{M}$ is the universal covering of M, Δ^q_{M,ρ_s} is the twisted Laplacian associated to the representation ρ_s, λ is any real number and $E_{eq}(\lambda, \Delta^q_{M,\rho_s})$ stands for the eigenforms of $\Delta^q_{\tilde{M}}$ associated to λ that are equivariant in relation to ρ_s.

So, to determine the eigenvalues and dimension of the eigenspaces of $\Delta^q_{\mathbb{R}\mathrm{P}^n}$ it is useful to identify among the eigenforms of $\Delta^q_{S^n}$ the ones that are equivariant in relation to the trivial representation σ of $\pi_1(\mathbb{R}\mathrm{P}^n)$ in $GL(\mathbb{R})$, since the twisted Laplacian $\Delta^q_{\mathbb{R}\mathrm{P}^n,\sigma}$ is essentially $\Delta^q_{\mathbb{R}\mathrm{P}^n}$. These equivariant eigenforms are precisely the pullbacks by the inclusion map $i : S^n \to \mathbb{R}^{n+1}$ of the harmonic polynomial forms $\omega \in C^\infty(\Lambda^q\mathbb{R}^{n+1})$ that satisfy $\omega(x) = \omega(-x)$ for every

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$x \in \mathbb{R}^{n+1}$. If H_k^q is one space of harmonic polynomial forms, then H_k^q contains forms with this property if, and only if, $q+k$ is even and in this case every element of H_k^q has this property. So, by Table 1 and (1) we obtain the decomposition of $C^\infty(\Lambda^q \mathbb{R}\mathrm{P}^n)$ into irreducible $SO(n+1)$-modules, which is summarized on Table 2. (The odd dimensional version of this table is present on [16]).

Table 2: Decomposition of the spaces of forms into $SO(n+1)$-irreducible modules on even dimensional projective spaces

$n = 2m$	
q	$C^\infty(\Lambda^q \mathbb{R}\mathrm{P}^n)$
$q = 0$	$\bigoplus_{k \geqslant 0} E(2k\Lambda_1)$
$1 \leqslant q \leqslant m-1;\ q$ even	$\bigoplus_{k \geqslant 0} E(2k\Lambda_1 + \Lambda_q) \oplus E((2k+1)\Lambda_1 + \Lambda_{q+1})$
$1 \leqslant q \leqslant m-1;\ q$ odd	$\bigoplus_{k \geqslant 0} E((2k+1)\Lambda_1 + \Lambda_q) \oplus E(2k\Lambda_1 + \Lambda_{q+1})$
$q = m$	$\bigoplus_{k \geqslant 0} E(2k\Lambda_1 + \Lambda_m) \oplus E((2k+1)\Lambda_1 + \Lambda_m)$
$m+1 \leqslant q \leqslant 2m-1;\ q$ even	$\bigoplus_{k \geqslant 0} E((2k+1)\Lambda_1 + \Lambda_{n-q}) \oplus E(2k\Lambda_1 + \Lambda_{n-q+1})$
$m+1 \leqslant q \leqslant 2m-1;\ q$ odd	$\bigoplus_{k \geqslant 0} E(2k\Lambda_1 + \Lambda_{n-q}) \oplus E((2k+1)\Lambda_1 + \Lambda_{n-q+1})$
$q = 2m$	$\bigoplus_{k \geqslant 0} E((2k+1)\Lambda_1)$

Now we just need to know the eigenvalue associated to each irreducible module and the dimension of this space. This subject was studied by Weng and You [22] when they calculated analytic torsion of spheres and is presented on Table 3, with the remark that their original work had a misprint that corresponds to the last line of the table when $q \neq m$ (see the appendix of [17] for a proof of this statement). We present the correct version of the data.

Table 3: Maximal weights, associated eigenvalues and dimensions of the $SO(n+1)$-irreducible modules

$n = 2m$		
$k\Lambda_1$	$k(k+n-1)$	$\frac{2k+2m-1}{2m-1}\binom{k+2m-2}{k}$
$k\Lambda_1 + \Lambda_q$ $2 \leqslant q \leqslant m$	$(k+q)(k+n+1-q)$	$\frac{2k+2m+1}{k+2m-q+1}\binom{k+2m}{k+q}\binom{k+q-1}{k}$

Beyond determining the spectrum and dimension of the eigenspaces of the Laplacian, we need some technical lemmas to simplify subsequent calculations. To establish these lemmas we will use the notation, with $q, j \in \{1, \dots, m\}$:

$$
\begin{gathered}
\mathbb{N} = \{1, 2, \dots\}, \quad \mathbb{N}_m := \{1, 2, \dots, m\}, \quad d_j^q = (j-q)(2m+1-q-j), \\
d^q = (d_l^q)_{l \in \mathbb{N}_m - \{q\}} := (d_1^q, \dots, \widehat{d_q^q}, \dots, d_m^q) \in \mathbb{R}^{m-1}, \\
w_2^q = (l-q)_{l \in \mathbb{N}_m - \{q\}} \times (2m+1-q-l)_{l \in \mathbb{N}_m - \{q\}}, \\
:= (1-q, \dots, \widehat{q-q}, \dots, m-q, 2m-q, \dots, 2m\widehat{+1-}2q, \dots, m+1-q) \in \mathbb{R}^{2m-2}
\end{gathered} \tag{2}
$$

and elementary symmetric polynomials $e_j(a_1, \dots, a_n)$, which are defined for complex numbers $a_1, \dots, a_n$ by the expression:

$$
\prod_{i=1}^{m} (x + a_i) = \sum_{j=0}^{m} e_{m-j}(a_1, \dots, a_m) x^j .
$$

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Lemma 2.1. *For $j \in \mathbb{N} \cup \{0\}$, defining:*

$$^{\dagger}\zeta_{\alpha,d,j}(s) = \sum_{k \geqslant 0} \frac{4k+2\alpha+1}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}},$$

with $d = 0$ and $\alpha \in \mathbb{N}$; or $d = 1$ and $\alpha \in \mathbb{N}\backslash\{1\}$, we have:

$$\begin{aligned}
^{\dagger}\zeta'_{\alpha,d,j}(0) &= \sum_{k=1}^{\alpha-1}(4k-2\alpha+1)\log(2k-d)(2k-d)^{j}(2k-2\alpha+1+d)^{j} \\
&+ \sum_{l}\left(\binom{j}{2l}+\binom{j+1}{2l}\right)(2\alpha-1-2d)^{2l}\zeta'_R(-2j+2l-1) \\
&+ \frac{1}{2}\sum_{l}\left(\binom{j}{2l+1}+\binom{j+1}{2l+1}\right)(2\alpha-1-2d)^{2l+1}(-1)^{1+d} \\
&\quad \times((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0} \\
&+ \frac{(-1)^{j+1}}{2}\frac{(j!)^2}{(2j+2)!}(2\alpha-1-2d)^{2j+2}.
\end{aligned}$$

Proof. For $\mathrm{Re}(s) \gg 0$:

$$\begin{aligned}
^{\dagger}\zeta'_{\alpha,d,j}(s) &= -\sum_{k\geqslant 0}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} - \sum_{k\geqslant 0}\frac{(4k+2\alpha+1)\log(2k+2\alpha-d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} \\
&= -\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} - \sum_{k\geqslant \alpha}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} \\
&\quad -\sum_{k\geqslant \alpha}\frac{(4k-2\alpha+1)\log(2k-d)}{(2k-2\alpha+1+d)^{s-j}(2k-d)^{s-j}} \\
&= -\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} \\
&\quad -\sum_{k\geqslant \alpha}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{2s-2j}}\left(1+\frac{2\alpha-1-2d}{2k+1+d}\right)^{-s+j} \\
&\quad -\sum_{k\geqslant \alpha}\frac{(4k-2\alpha+1)\log(2k-d)}{(2k-d)^{2s-2j}}\left(1-\frac{2\alpha-1-2d}{2k-d}\right)^{-s+j}
\end{aligned}$$

and using $\left|\frac{2\alpha-1-2d}{2k+1+d}\right| < 1$, $\left|\frac{2\alpha-1-2d}{2k-d}\right| < 1$ for $k \geqslant \alpha$ and the Binomial Theorem we have:

$$\begin{aligned}
^{\dagger}\zeta'_{\alpha,d,j}(s) &= -\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} \\
&\quad -\sum_{k\geqslant \alpha}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{2s-2j}}\sum_{l}\binom{-s+j}{l}\left(\frac{2\alpha-1-2d}{2k+1+d}\right)^{l} \\
&\quad -\sum_{k\geqslant \alpha}\frac{(4k-2\alpha+1)\log(2k-d)}{(2k-d)^{2s-2j}}\sum_{l}\binom{-s+j}{l}\left(-\frac{2\alpha-1-2d}{2k-d}\right)^{l}
\end{aligned}$$

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$$
\begin{aligned}
= \quad & -\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} - \sum_{l}\binom{-s+j}{l}(2\alpha-1-2d)^{l} \\
& \times\left[\sum_{k\geqslant 0}\frac{(2(2k+1+d)+(2\alpha-1-2d))\log(2k+1+d)}{(2k+1+d)^{2s-2j+l}}\right. \\
& \left.-\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{2s-2j+l}}\right] - \sum_{l}\binom{-s+j}{l}(-(2\alpha-1-2d))^{l} \\
& \times\left[\sum_{k\geqslant 1}\frac{(2(2k-d)-(2\alpha-1-2d))\log(2k-d)}{(2k-d)^{2s-2j+l}}\right. \\
& \left.-\sum_{k=1}^{\alpha-1}\frac{(4k-2\alpha+1)\log(2k-d)}{(2k-d)^{2s-2j+l}}\right] \\
= \quad & -\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} + \sum_{l}\binom{-s+j}{l} \\
& \times\left[(2\alpha-1-2d)^{l}\left(2^{-2s+2j-l+1}\zeta_H(2s-2j+l-1,\frac{1+d}{2})\right)'\right. \\
& +\frac{1}{2}(2\alpha-1-2d)^{l+1}\left(2^{-2s+2j-l}\zeta_H(2s-2j+l,\frac{1+d}{2})\right)' \\
& +(-(2\alpha-1-2d))^{l}\left(2^{-2s+2j-l+1}\zeta_H(2s-2j+l-1,\frac{2-d}{2})\right)' \\
& +\frac{1}{2}(-(2\alpha-1-2d))^{l+1}\left(2^{-2s+2j-l}\zeta_H(2s-2j+l,\frac{2-d}{2})\right)' \\
& +(2\alpha-1-2d)^{l}\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{2s-2j+l}} \\
& \left.+(-(2\alpha-1-2d))^{l}\sum_{k=1}^{\alpha-1}\frac{(4k-2\alpha+1)\log(2k-d)}{(2k-d)^{2s-2j+l}}\right] \\
= \quad & -\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{s-j}(2k+2\alpha-d)^{s-j}} + \sum_{l}\binom{-s+j}{l}w(\alpha,d,j,l) \\
& +\sum_{l}\binom{-s+j}{l}\left[(2\alpha-1-2d)^{l}\sum_{k=0}^{\alpha-1}\frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{2s-2j+l}}\right. \\
& \left.+(-(2\alpha-1-2d))^{l}\sum_{k=1}^{\alpha-1}\frac{(4k-2\alpha+1)\log(2k-d)}{(2k-d)^{2s-2j+l}}\right], \qquad (3)
\end{aligned}
$$

where in the last expression we are defining $w(\alpha,d,j,l)$.

To proceed with our study we need to analyse the behaviour of $\sum_{l}\binom{-s+j}{l}\cdot w(\alpha,d,j,l)$ as s tends to zero. This analysis will be divided in two parts, the first one considering the sum for $l \geq j+1$ and the second one considering it for $0 \leq l \leq j$.

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The Hurwitz zeta function $\zeta_H(s,\lambda)$; $\lambda \in (0,1]$ has a simple pole with residuum 1 in $s=1$ and is holomorphic in $\mathbb{C}\setminus\{1\}$. So the functions $(2^{-2s+2j-l+1}\zeta_H(2s-2j+l-1,\frac{1+d}{2}))'$ and $(2^{-2s+2j-l+1}\zeta_H(2s-2j+l-1,\frac{2-d}{2}))'$ have poles of order 2 in $s=j-\frac{l}{2}+1$ and are holomorphic in $\mathbb{C}\setminus\{j-\frac{l}{2}+1\}$. Their poles happen in $s=0$ if, and only if, $l=2j+2$. The functions $(2^{-2s+2j-l}\zeta_H(2s-2j+l,\frac{1+d}{2}))'$ and $(2^{-2s+2j-l}\zeta_H(2s-2j+l,\frac{2-d}{2}))'$ have poles of order 2 in $s=j-\frac{l}{2}+\frac{1}{2}$ and are holomorphic in $\mathbb{C}\setminus\{j-\frac{l}{2}+\frac{1}{2}\}$. Their poles happen in $s=0$ if, and only if, $l=2j+1$.

The power series expansions of $(2^{-x}\zeta_H(x,\frac{1+d}{2}))'$ and $(2^{-x}\zeta_H(x,\frac{2-d}{2}))'$ for $x=2s-2j+l-1$ or $x=2s-2j+l$ and for s tending to 0, when 0 is a pole are given by:

$$l=2j+2,\qquad \left(2^{-2s+2j-l+1}\zeta_H(2s-2j+l-1,\frac{1+d}{2})\right)'=-\frac{1}{4s^2}-\frac{\log 2}{2s}+a_0+O(s),$$

$$l=2j+2,\qquad \left(2^{-2s+2j-l+1}\zeta_H(2s-2j+l-1,\frac{2-d}{2})\right)'=-\frac{1}{4s^2}-\frac{\log 2}{2s}+b_0+O(s),$$

$$l=2j+1,\qquad \left(2^{-2s+2j-l}\zeta_H(2s-2j+l,\frac{1+d}{2})\right)'=-\frac{1}{4s^2}-\frac{\log 2}{2s}+c_0+O(s),$$

$$l=2j+1,\qquad \left(2^{-2s+2j-l}\zeta_H(2s-2j+l,\frac{2-d}{2})\right)'=-\frac{1}{4s^2}-\frac{\log 2}{2s}+d_0+O(s),$$

so for s next to zero:

$$\begin{aligned}
&\sum_{l\geqslant j+1}\binom{-s+j}{l}w(\alpha,d,j,l)\\
=\ &\frac{1}{2}\binom{-s+j}{2j+1}(2\alpha-1-2d)^{2j+2}\left(-\frac{1}{4s^2}-\frac{\log 2}{2s}+c_0+O(s)\right)\\
+\ &\frac{1}{2}\binom{-s+j}{2j+1}(-(2\alpha-1-2d))^{2j+2}\left(-\frac{1}{4s^2}-\frac{\log 2}{2s}+d_0+O(s)\right)\\
+\ &\binom{-s+j}{2j+2}(2\alpha-1-2d)^{2j+2}\left(-\frac{1}{4s^2}-\frac{\log 2}{2s}+a_0+O(s)\right)\\
+\ &\binom{-s+j}{2j+2}(-(2\alpha-1-2d))^{2j+2}\left(-\frac{1}{4s^2}-\frac{\log 2}{2s}+b_0+O(s)\right)\\
=\ &\left(\binom{-s+j}{2j+2}+\binom{-s+j+1}{2j+2}\right)(2\alpha-1-2d)^{2j+2}\left(-\frac{1}{4s^2}+O(s^{-1})\right)\\
=\ &\left(\frac{(-s+j)\dots(-s-j-1)}{(2j+2)!}+\frac{(-s+j+1)\dots(-s-j)}{(2j+2)!}\right)\\
&\times(2\alpha-1-2d)^{2j+2}(-\frac{1}{4s^2}+O(s^{-1}))\\
=\ &\left(\frac{(s-j)\dots(s+j+1)}{(2j+2)!}+\frac{(s-j-1)\dots(s+j)}{(2j+2)!}\right)(2\alpha-1-2d)^{2j+2}(-\frac{1}{4s^2}+O(s^{-1}))\\
=\ &\left(\frac{(s-j)\dots s\dots(s+j)(s+j+1+s-j-1)}{(2j+2)!}\right)(2\alpha-1-2d)^{2j+2}(-\frac{1}{4s^2}+O(s^{-1}))\\
=\ &-\frac{1}{2}\frac{(s-j)\dots(s-1)(s+1)\dots(s+j)}{(2j+2)!}(2\alpha-1-2d)^{2j+2}+O(s).
\end{aligned}$$

In the cases where $l\in\{0,\dots,j\}$ we just need to apply the expression that

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defines $\binom{-s+j}{l} \cdot w(\alpha, d, j, l)$ at $s = 0$. By using

$$\left(2^{-2t+2j-2l+1}\left(\zeta_H(2t-2j+2l-1, \frac{1+d}{2}) + \zeta_H(2t-2j+2l-1, \frac{2-d}{2})\right)\right)'|_{t=0} = 2\zeta_R'(-2j+2l-1)$$

and

$$\left(2^{-2t+2j-2l}(\zeta_H(2t-2j+2l, \frac{1+d}{2}) - \zeta_H(2t-2j+2l, \frac{2-d}{2}))\right)'|_{t=0} = (-1)^{1+d}((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0},$$

we obtain the following simplified expression for (3):

$$\begin{aligned}
{}^{\dagger}\zeta'_{\alpha,d,j}(s) &= -\sum_{k=0}^{\alpha-1} \frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{-j}(2k+2\alpha-d)^{-j}} \\
&+ \sum_l \Bigg[2\binom{j}{2l}(2\alpha-1-2d)^{2l}\zeta_R'(-2j+2l-1) \\
&+ \binom{j}{2l-1}(2\alpha-1-2d)^{2l}\zeta_R'(-2j+2l-1) \\
&+ \binom{j}{2l+1}(2\alpha-1-2d)^{2l+1}(-1)^{1+d}((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0} \\
&+ \frac{1}{2}\binom{j}{2l}(2\alpha-1-2d)^{2l+1}(-1)^{1+d}((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0}\Bigg] \\
&+ \sum_{k=0}^{\alpha-1} \frac{(4k+2\alpha+1)\log(2k+1+d)}{(2k+1+d)^{-2j}} \sum_l \binom{j}{l}\left(\frac{2\alpha-1-2d}{2k+1+d}\right)^l \\
&+ \sum_{k=1}^{\alpha-1} \frac{(4k-2\alpha+1)\log(2k-d)}{(2k-d)^{-2j}} \sum_l \binom{j}{l}\left(-\frac{2\alpha-1-2d}{2k-d}\right)^l \\
&+ \frac{(-1)^{j+1}}{2}\frac{(j!)^2}{(2j+2)!}(2\alpha-1-2d)^{2j+2} \\
&= \sum_{k=1}^{\alpha-1}(4k-2\alpha+1)\log(2k-d)(2k-d)^j(2k-2\alpha+1+d)^j \\
&+ \sum_l\left(\binom{j}{2l}+\binom{j+1}{2l}\right)(2\alpha-1-2d)^{2l}\zeta_R'(-2j+2l-1) \\
&+ \frac{1}{2}\sum_l\left(\binom{j}{2l+1}+\binom{j+1}{2l+1}\right)(2\alpha-1-2d)^{2l+1}(-1)^{1+d} \\
&\quad\times((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0} \\
&+ \frac{(-1)^{j+1}}{2}\frac{(j!)^2}{(2j+2)!}(2\alpha-1-2d)^{2j+2}
\end{aligned}$$

and this concludes the lemma. □

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Lemma 2.2. *Defining*

$$Z_q^*(x) = \sum_{j=0}^{m-1} \frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!} \sum_l \left(\binom{j}{2l+1} + \binom{j+1}{2l+1} \right) (2m+1-2q)^{2l+1} x^{2j-2l},$$

we have:

$$Z_q^*(x) = \frac{\sum_{k=0}^{m-1} ((2m+1-2q) e_{2m-2k-2}(w_2^q) + 2e_{2m-2k-1}(w_2^q)) x^{2k}}{(2m-q)!(q-1)!}.$$

Proof. Since

$$\begin{aligned}
&\sum_l \binom{j}{2l+1} (2m+1-2q)^{2l+1} x^{2j-2l} = x^{j+1} \sum_l \binom{j}{2l+1} (2m+1-2q)^{2l+1} x^{j-2l-1} \\
= \; & \frac{1}{2} x^{j+1} ((x + (2m+1-2q))^j - (x - (2m+1-2q))^j) \\
= \; & \frac{1}{2} x (x^2 + (2m+1-2q)x)^j - \frac{1}{2} x (x^2 - (2m+1-2q)x)^j
\end{aligned}$$

and

$$\begin{aligned}
&\sum_l \binom{j+1}{2l+1} (2m+1-2q)^{2l+1} x^{2j-2l} = x^j \sum_l \binom{j+1}{2l+1} (2m+1-2q)^{2l+1} x^{j+1-2l-1} \\
= \; & \frac{1}{2} x^j ((x + (2m+1-2q))^{j+1} - (x - (2m+1-2q))^{j+1}) \\
= \; & \frac{1}{2} (x + (2m+1-2q))(x^2 + (2m+1-2q)x)^j + \frac{1}{2}(-x + (2m+1-2q))(x^2 - (2m+1-2q)x)^j,
\end{aligned}$$

we have:

$$\begin{aligned}
Z_q^*(x) \; &= \; \frac{2x + 2m + 1 - 2q}{2(2m-q)!(q-1)!} \sum_{j=0}^{m-1} e_{m-j-1}(d^q)(x^2 + (2m+1-2q)x)^j \\
&+ \; \frac{-2x + 2m + 1 - 2q}{2(2m-q)!(q-1)!} \sum_{j=0}^{m-1} e_{m-j-1}(d^q)(x^2 - (2m+1-2q)x)^j
\end{aligned}$$

and using

$$\begin{aligned}
\sum_{j=0}^{m-1} e_{m-j-1}(d^q)(x^2 + (2m+1-2q)x)^j &= \prod_{j=1, j\neq q}^{m} (x + 2m + 1 - q - j)(x + j - q), \\
\sum_{j=0}^{m-1} e_{m-j-1}(d^q)(x^2 - (2m+1-2q)x)^j &= \prod_{j=1, j\neq q}^{m} (x - 2m - 1 + q + j)(x - j + q),
\end{aligned}$$

we obtain:

$$\begin{aligned}
Z_q^*(x) \; &= \; \frac{2x + 2m + 1 - 2q}{2(2m-q)!(q-1)!} \prod_{j=1, j\neq q}^{m} (x + 2m + 1 - q - j) \prod_{j=1, j\neq q}^{m} (x + j - q) \\
&+ \; \frac{-2x + 2m + 1 - 2q}{2(2m-q)!(q-1)!} \prod_{j=1, j\neq q}^{m} (x - (2m + 1 - q - j)) \prod_{j=1, j\neq q}^{m} (x - (j - q)),
\end{aligned} \tag{4}$$

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and therefore:

$$\begin{aligned}
Z_q^*(x) &= \frac{2x+2m+1-2q}{2(2m-q)!(q-1)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_2^q)x^{2k} \\
&+ \frac{2x+2m+1-2q}{2(2m-q)!(q-1)!}\sum_{k=0}^{m-1} e_{2m-2k-1}(w_2^q)x^{2k-1}
\end{aligned}$$

$$\begin{aligned}
&+ \frac{-2x+2m+1-2q}{2(2m-q)!(q-1)!}\sum_{k=0}^{m-1} e_{2m-2k-2}(w_2^q)x^{2k} \\
&- \frac{-2x+2m+1-2q}{2(2m-q)!(q-1)!}\sum_{k=0}^{m-1} e_{2m-2k-1}(w_2^q)x^{2k-1} \\
&= \frac{\sum_{k=0}^{m-1}((2m+1-2q)e_{2m-2k-2}(w_2^q)+2e_{2m-2k-1}(w_2^q))x^{2k}}{(2m-q)!(q-1)!},
\end{aligned}$$

with the convention $e_{2m-1}(w_2^q)=0$. □

We also need some lemmas which were proved by the author on [17]. We state these lemmas to introduce the associated notation.

Lemma 2.3 (Lemma 3.2 of [17])**.** *Let $n \in \mathbb{N}$ and let m be the largest integer less than or equal to $\frac{n+1}{2}$. For $q \in \mathbb{N}_m$ and $S = \{(j-q)(n+1-q-j) \mid j \in \mathbb{N}_m - \{q\}\}$ we have $S = S_1 \cup S_2$, where:*

$$S_1 = \{-k(k+n+1-2q) \mid 1 \leqslant k \leqslant q-1\},$$

$$S_2 = \{-k(k-n-1+2q) \mid 1 \leqslant k \leqslant n-2q\}.$$

Lemma 2.4 (Lemma 4.3 of [17])**.** *For every $q \in \mathbb{N}_m$ and $k \in \mathbb{N}\cup\{0\}$, let:*

$$\begin{aligned}
\xi^e_{0,k}(s) &= \dim E(k\Lambda_1)c^{-s}_{k\Lambda_1}, \\
\xi^e_{q,k}(s) &= \dim E(k\Lambda_1+\Lambda_q)c^{-s}_{k\Lambda_1+\Lambda_q}, k \in \mathbb{N}\cup\{0\},
\end{aligned} \tag{5}$$

where c_Λ denotes the eigenvalue associated to the maximal weight Λ. We have:

$$\xi^e_{q,k}(s) = \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\frac{2k+2m+1}{(k+q)^{s-j}(k+2m+1-q)^{s-j}}.$$

Lemma 2.5 (Lemma 4.4 of [17])**.** *Let*

$$Z_q(x) := 2\sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{l\geqslant 0}\left(\begin{pmatrix} j \\ 2l\end{pmatrix}+\begin{pmatrix} j+1 \\ 2l\end{pmatrix}\right)(2m+1-2q)^{2l}x^{2j-2l+2},$$

then:

$$Z_q(x) = \frac{4\sum_{k=0}^{m-1} e_{2m-2k-2}(w_2^q)x^{2k+2}+2(2m+1-2q)\sum_{k=0}^{m-2} e_{2m-2k-3}(w_2^q)x^{2k+2}}{(2m-q)!(q-1)!}.$$

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3 Determinant of the Laplacian on even dimensional real projective spaces

We proceed now with the construction of our determinant formulae. To obtain our formulae we will use the auxiliary functions $\xi^e_{q,k}(s)$ defined on Lemma 2.4. These functions are the same auxiliary functions defined in section 3.2 of [17] to calculate the regularized determinants of spheres. By using them and the decomposition of $C^\infty(\Lambda^q\mathbb{R}\mathrm{P}^{2m})$ into $SO(2m+1)$-irreducible modules given by Table 2 we may describe the spectral zeta functions of $\mathbb{R}\mathrm{P}^{2m}$ in the following way:

$$\begin{array}{rcll}
\zeta(s,\Delta^0_{\mathbb{R}\mathrm{P}^{2m}}) &=& \sum\limits_{k\geqslant 1}\xi^e_{0,2k}(s)=\sum\limits_{k\geqslant 0}\xi^e_{1,2k+1}(s) & \\
\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m}}) &=& \sum\limits_{k\geqslant 0}\xi^e_{q,2k}(s)+\xi^e_{q+1,2k+1}(s); & 1\leqslant q\leqslant m-1, \quad q \text{ even}\\
\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m}}) &=& \sum\limits_{k\geqslant 0}\xi^e_{q,2k+1}(s)+\xi^e_{q+1,2k}(s); & 1\leqslant q\leqslant m-1, \quad q \text{ odd}\\
\zeta(s,\Delta^m_{\mathbb{R}\mathrm{P}^{2m}}) &=& \sum\limits_{k\geqslant 0}\xi^e_{m,k}(s) & \\
\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m}}) &=& \sum\limits_{k\geqslant 0}\xi^e_{2m-q,2k+1}(s)+\xi^e_{2m-q+1,2k}(s); & m+1\leqslant q\leqslant 2m-1, \quad q \text{ even}\\
\zeta(s,\Delta^q_{\mathbb{R}\mathrm{P}^{2m}}) &=& \sum\limits_{k\geqslant 0}\xi^e_{2m-q,2k}(s)+\xi^e_{2m-q+1,2k+1}(s); & m+1\leqslant q\leqslant 2m-1, \quad q \text{ odd}\\
\zeta(s,\Delta^{2m}_{\mathbb{R}\mathrm{P}^{2m}}) &=& \sum\limits_{k\geqslant 0}\xi^e_{0,2k+1}(s)=\sum\limits_{k\geqslant 0}\xi^e_{1,2k}(s). &
\end{array} \tag{6}$$

So, to determine the derivatives at zero of the spectral zeta functions we just need to determine the derivatives at zero of the functions $\sum\limits_{k\geqslant 0}\xi^e_{q,2k}(s)$ and $\sum\limits_{k\geqslant 0}\xi^e_{q,2k+1}(s)$ for $q\in\mathbb{N}_m$.

If $q\in\mathbb{N}_m$ and $\phi\in\{0,1\}$ satisfy $q=2a+b$ with $b\in\{0,1\}$ and $b+\phi=1$, then by Lemma 2.4:

$$\begin{aligned}
\sum_{k\geqslant 0}\xi^e_{q,2k+\phi}(s) &= \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k\geq 0}\frac{4k+2\phi+2m+1}{(2k+\phi+q)^{s-j}(2k+\phi+2m+1-q)^{s-j}}\\
&= \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k\geqslant 0}\frac{4k+2m+3-2q}{(2k+1)^{s-j}(2k+2m+2-2q)^{s-j}}\\
&- \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k=0}^{a-1}\frac{4k+2m+3-2q}{(2k+1)^{s-j}(2k+2m+2-2q)^{s-j}}.
\end{aligned}$$

Since $0\leqslant k\leqslant a-1$ implies $1\leqslant 2k+1\leqslant q-1$ we may apply part S_1 of Lemma 2.3 to obtain:

$$\sum_{k=0}^{a-1}\sum_{j=0}^{m-1}e_{m-j-1}(d^q)(2k+1)^j(2k+2m+2-2q)^j=0$$

and then simplify the expression of $\sum\limits_{k\geqslant 0}\xi^e_{q,2k+\phi}(s)$ to:

$$\sum_{k\geqslant 0}\xi^e_{q,2k+\phi}(s)=\sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k\geqslant 0}\frac{4k+2m+3-2q}{(2k+1)^{s-j}(2k+2m+2-2q)^{s-j}},$$

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Therefore, by Lemma 2.1 with $\alpha = m+1-q$ and $d=0$ we obtain:

$$\left(\sum_{k\geqslant 0}\xi^e_{q,2k+\phi}(s)\right)'|_{s=0} = \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}$$
$$\times\left[\sum_{k=0}^{m-1-q}(4k-2m+3+2q)\log(2k+2)(2k+2)^j(2k-2m+1+2q)^j\right.$$
$$+\sum_l\left(\binom{j}{2l}+\binom{j+1}{2l}\right)(2m+1-2q)^{2l}\zeta_R'(-2j+2l-1)$$
$$-\frac{1}{2}\sum_l\left(\binom{j}{2l+1}+\binom{j+1}{2l+1}\right)(2m+1-2q)^{2l+1}((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0}$$
$$\left.+\frac{(-1)^{j+1}}{2}\frac{(j!)^2}{(2j+2)!}(2m+1-2q)^{2j+2}\right].$$

If $q\in\mathbb{N}_m$ and $\phi\in\{0,1\}$ satisfy $q=2a+b$ with $b\in\{0,1\}$ and $b+\phi\in\{0,2\}$, then by Lemma 2.4:

$$\begin{aligned}\sum_{k\geqslant 0}\xi^e_{q,2k+\phi}(s) &= \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k\geq 0}\frac{4k+2\phi+2m+1}{(2k+\phi+q)^{s-j}(2k+\phi+2m+1-q)^{s-j}}\\ &= \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k\geqslant 0}\frac{4k+2m+5-2q}{(2k+2)^{s-j}(2k+2m+3-2q)^{s-j}}\\ &- \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k=0}^{a+\phi-2}\frac{4k+2m+5-2q}{(2k+2)^{s-j}(2k+2m+3-2q)^{s-j}}\end{aligned}$$

and since $0\leqslant k\leqslant a+\phi-2$ implies $0\leqslant 2k+\phi\leqslant q-1$ we may apply part S_1 of Lemma 2.3 to obtain:

$$\sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k=0}^{a+\phi-2}\frac{4k+2m+5-2q}{(2k+2)^{s-j}(2k+2m+3-2q)^{s-j}}=0$$

and then simplify the expression of $\sum_{k\geqslant 0}\xi^e_{q,2k+\phi}(s)$ to:

$$\sum_{k\geqslant 0}\xi^e_{q,2k+\phi}(s) = \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k\geqslant 0}\frac{4k+2m+5-2q}{(2k+2)^{s-j}(2k+2m+3-2q)^{s-j}}.$$

Therefore, an application of Lemma 2.1 with $\alpha=m+2-q$ and $d=1$ yields:

$$\left(\sum_{k\geqslant 0}\xi^e_{q,2k+\phi}(s)\right)'|_{s=0} = \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}$$
$$\times\left[\sum_{k=0}^{m-q}(4k-2m+1+2q)\log(2k+1)(2k+1)^j(2k-2m+2q)^j\right.$$
$$+\sum_l\left(\binom{j}{2l}+\binom{j+1}{2l}\right)(2m+1-2q)^{2l}\zeta_R'(-2j+2l-1)$$

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$$+\frac{1}{2}\sum_{l}\left(\begin{pmatrix} j \\ 2l+1 \end{pmatrix}+\begin{pmatrix} j+1 \\ 2l+1 \end{pmatrix}\right)(2m+1-2q)^{2l+1}((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0}$$
$$\left.+\frac{(-1)^{j+1}}{2}\frac{(j!)^2}{(2j+2)!}(2m+1-2q)^{2j+2}\right].$$

It is possible to simplify the expression of $\left(\sum_{k\geqslant 0}\xi^{e}_{q,2k+\phi}(s)\right)'|_{s=0}$. Defining:

$$\begin{aligned}
Z_1^q &= \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k=0}^{m-q}(4k-2m+1+2q)\log(2k+1)(2k+1)^j(2k-2m+2q)^j \\
Z_1^{q\prime} &= \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{k=0}^{m-q-1}(4k-2m+3+2q)\log(2k+2)(2k+2)^j(2k-2m+1+2q)^j \\
Z_2^q &= \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{l}\left(\begin{pmatrix} j \\ 2l \end{pmatrix}+\begin{pmatrix} j+1 \\ 2l \end{pmatrix}\right)(2m+1-2q)^{2l}\zeta_R'(-2j+2l-1) \\
Z_3^q &= \frac{1}{2}\sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_{l}\left(\begin{pmatrix} j \\ 2l+1 \end{pmatrix}+\begin{pmatrix} j+1 \\ 2l+1 \end{pmatrix}\right)(2m+1-2q)^{2l+1} \\
&\quad ((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0} \\
Z_4^q &= \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\frac{(-1)^{j+1}}{2}\frac{(j!)^2}{(2j+2)!}(2m+1-2q)^{2j+2},
\end{aligned}$$

we have for $b+\phi\in\{0,2\}$:

$$\left(\sum_{k\geqslant 0}\xi^{e}_{q,2k+\phi}(s)\right)'|_{s=0}=Z_1^q+Z_2^q+Z_3^q+Z_4^q, \tag{7}$$

and for $b+\phi=1$:

$$\left(\sum_{k\geqslant 0}\xi^{e}_{q,2k+\phi}(s)\right)'|_{s=0}=Z_1^{q\prime}+Z_2^q-Z_3^q+Z_4^q. \tag{8}$$

The following reasoning will simplify the expressions of Z_1^q and $Z_1^{q\prime}$:

$$\forall r\in\mathbb{N}_{2m-2q+1}:\ \sum_{j=0}^{m-1}e_{m-j-1}(d^q)r^j(r-2m-1+2q)^j=\prod_{j=1,j\neq q}^{m}(r(r-2m-1+2q)+d_j^q)$$

and this product is equal to zero for $r\in\mathbb{N}_{2m-2q}$ by part S_2 of Lemma 2.3, and equal to $\prod_{j=1,j\neq q}^{m}d_j^q$ for $r=2m+1-2q$. Therefore:

$$\begin{aligned}
Z_1^q &= \frac{(2m+1-2q)\log(2m+1-2q)}{(2m-q)!(q-1)!}\prod_{j=1,j\neq q}^{m}(j-q)(2m+1-q-j) \\
&= \frac{\log(2m+1-2q)}{(2m-q)!(q-1)!}(-1)^{q-1}(q-1)!(2m-q)!=(-1)^{q-1}\log(2m+1-2q)
\end{aligned}$$

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and

$Z_1^{q\prime} = 0.$

About Z_2^q, we may substitute $\zeta_R'(-2j+2l-1)$ by $x^{2j-2l+2}$, apply Lemma 2.5 and substitute back x^{2k+2} by $\zeta_R'(-2k-1)$ to obtain:

$$Z_2^q = \frac{\sum_{k=0}^{m-1}(2e_{2m-2k-2}(w_2^q)+(2m+1-2q)e_{2m-2k-3}(w_2^q))\zeta_R'(-2k-1)}{(2m-q)!(q-1)!},$$

with the convention $e_{-1}(w_2^q)=0$.

About Z_3^q, substituting

$$((2^{-2t+2j-2l+1}-1)\zeta_R(2t-2j+2l))'|_{t=0} \quad \text{by} \quad 2x^{2j-2l},$$

we obtain the polynomial

$$Z_q^*(x) = \sum_{j=0}^{m-1}\frac{e_{m-j-1}(d^q)}{(2m-q)!(q-1)!}\sum_l\left(\begin{pmatrix} j \\ 2l+1\end{pmatrix}+\begin{pmatrix} j+1 \\ 2l+1\end{pmatrix}\right)(2m+1-2q)^{2l+1}x^{2j-2l},$$

which was studied on Lemma 2.2 and satisfies:

$$Z_q^*(x) = \frac{\sum_{k=0}^{m-1}((2m+1-2q)e_{2m-2k-2}(w_2^q)+2e_{2m-2k-1}(w_2^q))x^{2k}}{(2m-q)!(q-1)!}.$$

Therefore, substituting back $2x^{2k}$ by $((2^{-2t+2k+1}-1)\zeta_R(2t-2k))'|_{t=0}$, we have:

$$Z_3^q = \frac{\sum_{k=0}^{m-1}((2m+1-2q)e_{2m-2k-2}(w_2^q)+2e_{2m-2k-1}(w_2^q))\left((2^{-2t+2k+1}-1)\zeta_R(2t-2k)\right)'|_{t=0}}{2(2m-q)!(q-1)!}$$

and using the values of $\zeta_R(-2r);\ r\in\mathbb{N}\cup\{0\}$ to produce the equalities

$$\begin{aligned} ((2^{-2t+1}-1)\zeta_R(2t))'|_{t=0} &= 2\log 2+2\zeta_R'(0),\\ \forall k\in\mathbb{N}:\quad ((2^{-2t+2k+1}-1)\zeta_R(2t-2k))'|_{t=0} &= 2(2^{2k+1}-1)\zeta_R'(-2k), \end{aligned} \tag{9}$$

we obtain the following simplified expression for Z_3^q:

$$Z_3^q = \sum_{k=0}^{m-1}\frac{((2m+1-2q)e_{2m-2k-2}(w_2^q)+2e_{2m-2k-1}(w_2^q))(2^{2k+1}-1)\zeta_R'(-2k)}{(2m-q)!(q-1)!} + (-1)^{q-1}\log 2$$

By using (6), (7), (8), Theorem 4.2 of [17] and the simplified expressions for Z_1^q, $Z_1^{q\prime}$, Z_2^q and Z_3^q we obtain the derivatives at zero of the espectral zeta functions of $\Delta^q_{\mathbb{RP}^{2m}}$, and consequently, the determinants of the Laplacian on forms over even dimensional projective spaces.

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Theorem 3.1. *The derivatives at zero of the spectral zeta functions of the real even dimensional projective spaces are given by:*

$$\begin{aligned}
\zeta'(0,\Delta^0_{\mathbb{RP}^{2m}}) &= \frac{1}{2}\zeta'(0,\Delta^0_{S^{2m}}) + \frac{1}{2}\log(2m-1) + \log 2\\
&+ \sum_{k=0}^{m-1} \frac{((2m-1)e_{2m-2k-2}(w_2^1) + 2e_{2m-2k-1}(w_2^1))}{(2m-1)!}(2^{2k+1}-1)\zeta_R'(-2k),\\
\zeta'(0,\Delta^q_{\mathbb{RP}^{2m}}) &= \frac{1}{2}\zeta'(0,\Delta^q_{S^{2m}}) + \frac{(-1)^q}{2}\log\left(\frac{2m-1-2q}{2m+1-2q}\right)\\
&+ \sum_{k=0}^{m-1} \frac{((2m+1-2q)e_{2m-2k-2}(w_2^q) + 2e_{2m-2k-1}(w_2^q))}{(2m-q)!(q-1)!}(2^{2k+1}-1)\zeta_R'(-2k)\\
&+ \sum_{k=0}^{m-1} \Bigg[\frac{((2m-1-2q)e_{2m-2k-2}(w_2^{q+1}) + 2e_{2m-2k-1}(w_2^{q+1}))}{(2m-q-1)!q!}\\
&\times(2^{2k+1}-1)\zeta_R'(-2k)\Bigg];\ 1\leqslant q\leqslant m-1,\\
\zeta'(0,\Delta^m_{\mathbb{RP}^{2m}}) &= \frac{1}{2}\zeta'(0,\Delta^m_{S^{2m}}),\\
\zeta'(0,\Delta^q_{\mathbb{RP}^{2m}}) &= \frac{1}{2}\zeta'(0,\Delta^{2m-q}_{S^{2m}}) - \frac{(-1)^q}{2}\log\left(\frac{-2m-1+2q}{-2m+1+2q}\right)\\
&- \sum_{k=0}^{m-1} \Bigg[\frac{((-2m+1+2q)e_{2m-2k-2}(w_2^{2m-q}) + 2e_{2m-2k-1}(w_2^{2m-q}))}{q!(2m-q-1)!}(2^{2k+1}-1)\times\zeta_R'(-2k)\Bigg]\\
&- \sum_{k=0}^{m-1} \Bigg[\frac{((-2m-1+2q)e_{2m-2k-2}(w_2^{2m-q+1}) + 2e_{2m-2k-1}(w_2^{2m-q+1}))}{(q-1)!(2m-q)!}\\
&\times(2^{2k+1}-1)\zeta_R'(-2k)\Bigg];\ m+1\leqslant q\leqslant 2m-1,\\
\zeta'(0,\Delta^{2m}_{\mathbb{RP}^{2m}}) &= \frac{1}{2}\zeta'(0,\Delta^0_{S^{2m}}) - \frac{1}{2}\log(2m-1) - \log 2\\
&- \sum_{k=0}^{m-1} \frac{((2m-1)e_{2m-2k-2}(w_2^1) + 2e_{2m-2k-1}(w_2^1))}{(2m-1)!}(2^{2k+1}-1)\zeta_R'(-2k),
\end{aligned}$$

where w_2^q is defined by (2) and we adopt the convention $e_{-1}(w_2^q) = e_{2m-1}(w_2^q) = 0$.

An interesting corollary of this theorem is the relation between the derivatives at zero of the spectral zeta functions of even dimensional spheres and projective spaces.

Corollary 3.2. *For $m\in\mathbb{N}$ and $p\in\mathbb{N}_m\cup\{0\}$, the derivatives at zero of the spectral zeta functions of the Laplacian on forms over spheres and projective spaces satisfy the following property:*

$$\zeta'(0,\Delta^{m-p}_{S^{2m}}) = \zeta'(0,\Delta^{m-p}_{\mathbb{RP}^{2m}}) + \zeta'(0,\Delta^{m+p}_{\mathbb{RP}^{2m}}) = \zeta'(0,\Delta^{m+p}_{S^{2m}}).$$

Proof. This result follows from direct comparison of Theorem 3.1 above and Theorem 4.2 of [17]. □

We may also use Theorem 3.1 to calculate the regularized determinants of the twisted Laplacians $\det\Delta^p_{\mathbb{RP},\rho}$ for orthogonal or unitary representations $\rho:\pi_1(\mathbb{RP}^{2m})\to\mathrm{GL}(E)$.

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Corollary 3.3. *Let $\rho : \pi_1(\mathbb{R}\mathrm{P}^{2m}) \to \mathrm{GL}(E)$ be an orthogonal or unitary representation of $\pi_1(\mathbb{R}\mathrm{P}^{2m})$ and let*

$$\rho \simeq \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{p} \oplus \underbrace{\sigma_2 \oplus \cdots \oplus \sigma_2}_{N-p}.$$

be a decomposition of ρ into irreducible representations, where σ_1 is the trivial irreducible representation. The determinant of $\Delta^q_{\mathbb{R}\mathrm{P}^{2m},\rho}$, $q \in \{0, \dots, 2m\}$ satisfies:

$$\det \Delta^q_{\mathbb{R}\mathrm{P}^{2m},\rho} \quad = \quad \left(\det \Delta^q_{\mathbb{R}\mathrm{P}^{2m}}\right)^{2p-N} \cdot \left(\det \Delta^q_{S^{2m}}\right)^{N-p}.$$

Proof. By the decomposition of ρ into irreducible representations we have:

$$\det \Delta^q_{\mathbb{R}\mathrm{P}^{2m},\rho} \quad = \quad \left(\det \Delta^q_{\mathbb{R}\mathrm{P}^{2m},\sigma_1}\right)^p \cdot \left(\det \Delta^q_{\mathbb{R}\mathrm{P}^{2m}_{\sigma_2}}\right)^{N-p},$$

because given two equivalent representations $\rho_1 : \pi_1(M) \to E$ and $\rho_2 : \pi_1(M) \to E$ we have $\det \Delta^q_{M,\rho_1} = \det \Delta^q_{M,\rho_2}$ for every $q \in \{0, \dots, \dim M\}$.

Since $\det \Delta^q_{\mathbb{R}\mathrm{P}^{2m},\sigma_1} = \det \Delta^q_{\mathbb{R}\mathrm{P}^{2m}}$ and

$$\begin{aligned} \forall \lambda \in [0, +\infty), \quad & E(\lambda, \Delta^q_{S^{2m}}) = E(\lambda, \Delta^q_{\mathbb{R}\mathrm{P}^{2m},\sigma_1}) \oplus E(\lambda, \Delta^q_{\mathbb{R}\mathrm{P}^{2m},\sigma_2}) \\ & \Rightarrow \det \Delta^q_{\mathbb{R}\mathrm{P}^{2m},\sigma_2} = \left(\det \Delta^q_{S^{2m}}\right) \cdot \left(\det \Delta^q_{\mathbb{R}\mathrm{P}^{2m},\sigma_1}\right)^{-1}, \end{aligned}$$

the result follows. ☐

Note that our formulae for determinants generalize the previous formula on even dimensional projective spaces, see [8], since that formula could only be applied to $\Delta^0_{\mathbb{R}\mathrm{P}^{2m}}$. For a similar formula in the odd dimensional case see [16].

4 Analytic torsion

By using the definition of analytic torsion and Theorem 3.1, we are able to calculate the analytic torsion of even dimensional projective spaces. For $R_k = (2^{2k+1} - 1)\zeta'_R(-2k)$:

$$\begin{aligned} & 2\log(T_a(\mathbb{R}\mathrm{P}^{2m})) = \sum_{q=0}^{2m} (-1)^q q \zeta'(0, \Delta^q_{\mathbb{R}\mathrm{P}^{2m}}) \\ = \quad & \sum_{q=1}^{m-1} (-1)^q q \left[\frac{1}{2}\zeta'(0, \Delta^q_{S^{2m}}) + \frac{(-1)^q}{2} \log\left(\frac{2m-1-2q}{2m+1-2q}\right) \right. \\ + \quad & \sum_{k=0}^{m-1} \frac{((2m+1-2q)e_{2m-2k-2}(w_2^q) + 2e_{2m-2k-1}(w_2^q))}{(2m-q)!(q-1)!} R_k \\ + \quad & \left. \sum_{k=0}^{m-1} \frac{((2m-1-2q)e_{2m-2k-2}(w_2^{q+1}) + 2e_{2m-2k-1}(w_2^{q+1}))}{(2m-q-1)!q!} R_k \right] \\ + \quad & (-1)^m m \left[\frac{1}{2}\zeta'(0, \Delta^m_{S^{2m}})\right] \\ + \quad & \sum_{q=m+1}^{2m-1} (-1)^q q \left[\frac{1}{2}\zeta'(0, \Delta^{2m-q}_{S^{2m}}) - \frac{(-1)^q}{2} \log\left(\frac{-2m-1+2q}{-2m+1+2q}\right) \right. \end{aligned}$$

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$$
\begin{aligned}
&- \sum_{k=0}^{m-1} \frac{((-2m+1+2q)e_{2m-2k-2}(w_2^{2m-q}) + 2e_{2m-2k-1}(w_2^{2m-q}))}{q!(2m-q-1)!} R_k \\
&- \left. \sum_{k=0}^{m-1} \frac{((-2m-1+2q)e_{2m-2k-2}(w_2^{2m-q+1}) + 2e_{2m-2k-1}(w_2^{2m-q+1}))}{(q-1)!(2m-q)!} R_k \right] \\
&+ (-1)^{2m}(2m) \left[\frac{1}{2}\zeta'(0, \Delta_{S^{2m}}^{2m}) - \frac{1}{2}\log(2m-1) - \log 2 \right. \\
&- \left. \sum_{k=0}^{m-1} \frac{((2m-1)e_{2m-2k-2}(w_2^1) + 2e_{2m-2k-1}(w_2^1))}{(2m-1)!} R_k \right].
\end{aligned}
$$

So, defining:

$$
\begin{aligned}
T_1 &= \sum_{q=0}^{2m} (-1)^q q \left[\frac{1}{2}\zeta'(0, \Delta_{S^{2m}}^q) \right], \\
T_2 &= \sum_{q=1}^{m-1} \frac{q}{2} \left[\log\left(\frac{2m-1-2q}{2m+1-2q} \right) \right] - \sum_{q=m+1}^{2m-1} \frac{q}{2} \left[\log\left(\frac{-2m-1+2q}{-2m+1+2q} \right) \right] - m\log(2m-1), \\
T_3 &= \sum_{q=1}^{m-1} (-1)^q q \left[\sum_{k=0}^{m-1} \frac{((2m+1-2q)e_{2m-2k-2}(w_2^q) + 2e_{2m-2k-1}(w_2^q))}{(2m-q)!(q-1)!} R_k \right. \\
&+ \left. \sum_{k=0}^{m-1} \frac{((2m-1-2q)e_{2m-2k-2}(w_2^{q+1}) + 2e_{2m-2k-1}(w_2^{q+1}))}{(2m-q-1)!q!} R_k \right] \\
&+ \sum_{q=m+1}^{2m-1} (-1)^q q \left[- \sum_{k=0}^{m-1} \frac{((-2m+1+2q)e_{2m-2k-2}(w_2^{2m-q}) + 2e_{2m-2k-1}(w_2^{2m-q}))}{q!(2m-q-1)!} R_k \right. \\
&- \left. \sum_{k=0}^{m-1} \frac{((-2m-1+2q)e_{2m-2k-2}(w_2^{2m-q+1}) + 2e_{2m-2k-1}(w_2^{2m-q+1}))}{(q-1)!(2m-q)!} R_k \right] \\
&+ (2m) \left[- \sum_{k=0}^{m-1} \frac{((2m-1)e_{2m-2k-2}(w_2^1) + 2e_{2m-2k-1}(w_2^1))}{(2m-1)!} R_k - \log 2 \right],
\end{aligned}
$$

we obtain:

$$
2\log T_a(\mathbb{R}\mathrm{P}^{2m}) = T_1 + T_2 + T_3, \tag{10}
$$

where, by Theorem 2.3 of [19] and the definition of analytic torsion we obtain for T_1:

$$
T_1 = \log(T_a(S^{2m})) = 0. \tag{11}
$$

About T_2, it's expression may be simplified to:

$$
\begin{aligned}
T_2 &= \sum_{q=1}^{m-1} q \left[\frac{1}{2}\log(2m-1-2q) - \frac{1}{2}\log(2m+1-2q) \right] - m\log(2m-1) \\
&+ \sum_{q=m+1}^{2m-1} q \left[-\frac{1}{2}\log(-2m-1+2q) + \frac{1}{2}\log(-2m+1+2q) \right]
\end{aligned}
$$

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$$
\begin{aligned}
&= \sum_{q=1}^{m-1}\left[-\frac{1}{2}\log(2m+1-2q)\right]+\sum_{q=m+1}^{2m-1}\left[-\frac{1}{2}\log(-2m-1+2q)\right] \\
&+ \frac{2m-1}{2}\log(2m-1)-m\log(2m-1) \\
&= \sum_{q=1}^{m-1}\left[-\frac{1}{2}\log(2m+1-2q)\right]+\sum_{q=1}^{m-1}\left[-\frac{1}{2}\log(2m-1-2q)\right]-\frac{1}{2}\log(2m-1) \\
&= -\sum_{q=1}^{m-1}\log(2m+1-2q)=-\sum_{q=1}^{m}\log(2q-1). \qquad (12)
\end{aligned}
$$

And about T_3, using the notation of the previous section, we have

$$
T_3=\sum_{q=1}^{m-1}(-1)^q q\left[Z_3^q+Z_3^{q+1}\right]+\sum_{q=m+1}^{2m-1}(-1)^q q\left[-Z_3^{2m-q}-Z_3^{2m-q+1}\right]-2m\left[Z_3^1\right],
$$

and thus, substituting $\left((2^{-2t+2k+1}-1)\zeta_R(2t-2k)\right)'|_{t=0}$ by $2x^{2k}$, as we did in the previous section, we obtain

$$
\begin{aligned}
T_3(x) &:= \sum_{q=1}^{m-1}(-1)^q q\left[Z_q^*(x)+Z_{q+1}^*(x)\right] \\
&+ \sum_{q=m+1}^{2m-1}(-1)^q q\left[-Z_{2m-q}^*(x)-Z_{2m-q+1}^*(x)\right]-2m\left[Z_1^*(x)\right] \\
&= \sum_{q=1}^{m-1}(-1)^q q\left[Z_q^*(x)+Z_{q+1}^*(x)\right]+(-1)^{2m}(2m)\left[-Z_1^*(x)\right] \\
&+ \sum_{q=1}^{m-1}(-1)^{2m-q}(2m-q)\left[-Z_q^*(x)-Z_{q+1}^*(x)\right] \\
&= 2\sum_{q=1}^{m}(-1)^q Z_q^*(x)
\end{aligned}
$$

so, by (4):

$$
\begin{aligned}
(2m-1)!T_3(x) &= \sum_{q=1}^{m}(-1)^q\begin{pmatrix}2m-1\\ q-1\end{pmatrix}(2x+2m+1-2q) \\
&\times \prod_{j=1,j\neq q}^{m}(x+2m+1-q-j)(x+j-q) \\
&+ \sum_{q=1}^{m}(-1)^q\begin{pmatrix}2m-1\\ q-1\end{pmatrix}(-2x+2m+1-2q) \\
&\times \prod_{j=1,j\neq q}^{m}(x-2m-1+q+j)(x-j+q).
\end{aligned}
$$

Defining $p=2m+1-q$, we obtain that for q varying in $\{1,\ldots,m\}$, p varies in $\{m+1,\ldots,2m\}$ and moreover:

$$
\forall j\in\mathbb{N}_m-\{q\}:\quad x-2m-1+q+j=x+j-p \quad\text{and}\quad x-j+q=x+2m+1-p-j,
$$

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so that:

$$
\begin{aligned}
(2m-1)!T_3(x) &= \sum_{q=1}^{m}(-1)^q \begin{pmatrix} 2m-1 \\ q-1 \end{pmatrix} (2x+2m+1-2q) \\
&\times \prod_{j=1, j\neq q}^{m} (x+2m+1-q-j)(x+j-q) \\
&+ \sum_{p=m+1}^{2m} (-1)^{2m+1-p} \begin{pmatrix} 2m-1 \\ 2m-p \end{pmatrix} (-2x-2m-1+2p) \\
&\times \prod_{j=1, j\neq 2m+1-p}^{m} (x+j-p)(x+2m+1-p-j).
\end{aligned}
$$

Observe that if x is a natural number, $x = k$:

$$
\begin{aligned}
\forall q \in \mathbb{N}_m : \quad & \prod_{j=1, j\neq q}^{m} (k+2m+1-q-j)(k+j-q) \\
&= \frac{(k+2m-q)\ldots(k+m+1-q)}{(k+2m+1-2q)} \frac{(k+m-q)\ldots(k+1-q)}{k} \\
&= \frac{(k+2m-q)!}{(k-q)!}\left(\frac{1}{k(k+2m+1-2q)}\right)
\end{aligned}
$$

and for every $p \in \{m+1, \ldots, 2m\}$, we have the equalities:

$$
\begin{pmatrix} 2m-1 \\ 2m-p \end{pmatrix} = \begin{pmatrix} 2m-1 \\ p-1 \end{pmatrix},
$$

$$
\prod_{j=1, j\neq 2m+1-p}^{m} (k+j-p)(k+2m+1-p-j) = \frac{(k+2m-p)!}{(k-p)!}\left(\frac{1}{k(k+2m+1-2p)}\right).
$$

Therefore $(2m-1)!T_3(k)$ may be expressed as:

$$
\begin{aligned}
(2m-1)!T_3(k) &= \sum_{q=1}^{2m}(-1)^q \begin{pmatrix} 2m-1 \\ q-1 \end{pmatrix} \frac{(k+2m-q)!}{(k-q)!}(2k+2m+1-2q) \\
&\times \frac{1}{k(k+2m+1-2q)} \\
\Rightarrow T_3(k) &= \sum_{q=1}^{2m}(-1)^q \begin{pmatrix} 2m-1 \\ q-1 \end{pmatrix} \frac{(k+2m-q)!}{(k-q)!(2m)!} \cdot 2m \\
&\times \left(\frac{k}{k(k+2m+1-2q)} + \frac{k+2m+1-2q}{k(k+2m+1-2q)}\right) \\
\Rightarrow T_3(k) &= 2m\sum_{q=1}^{2m}(-1)^q \begin{pmatrix} 2m-1 \\ q-1 \end{pmatrix} \begin{pmatrix} k+2m-q \\ 2m \end{pmatrix} \\
&\times \left(\frac{1}{k+2m+1-2q} + \frac{1}{k}\right)
\end{aligned}
$$

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$$\begin{aligned}\Rightarrow T_3(k) &= -2m \sum_{q=0}^{2m-1} (-1)^q \binom{2m-1}{q} \binom{k+2m-1-q}{2m} \\ &\times \left(\frac{1}{k+2m-1-2q} + \frac{1}{k} \right)\end{aligned}$$

and defining $r := 2m-1-q$, what implies $q = 2m-1-r$, we obtain:

$$T_3(k) = 2m \sum_{r=0}^{2m-1} (-1)^r \binom{2m-1}{r} \binom{k+r}{2m} \left(\frac{1}{k+2r-(2m-1)} + \frac{1}{k} \right).$$

Now, applying Lemma 3.3 of [22], we obtain:

$$T_3(k) = -2m.$$

Hence $T_3(x)$ is a polynomial that satisfies $T_3(k) = -2m$ for every $k \in \mathbb{N}$, and we conclude $T_3(x)$ is the constant polynomial $T_3(x) = -2m$.

Therefore, substituting back $2x^{2k}$ by $\left((2^{-2t+2k+1}-1)\zeta_R(2t-2k)\right)'|_{t=0}$ we obtain for T_3 :

$$T_3 = -m((2^{-2t+1}-1)\zeta_R(2t))'|_{t=0} = -2m\log 2 + m\log(2\pi). \tag{13}$$

Expressions (10), (11), (12) and (13) imply that:

$$T_a(\mathbb{R}\mathrm{P}^{2m}) = \exp\left(\frac{1}{2}(T_1+T_2+T_3)\right) = \frac{1}{\sqrt{\prod_{q=1}^{m}(2q-1)}} \frac{\sqrt{(2\pi)^m}}{2^m} = \frac{\pi^{\frac{m}{2}}\sqrt{m!}}{\sqrt{(2m)!}},$$

what proves the following Theorem (see [2] for the volume of $\mathbb{R}\mathrm{P}^{2m}$):

Theorem 4.1. *The analytic torsion of even dimensional projective spaces is given by:*

$$T_a(\mathbb{R}\mathrm{P}^{2m}) = \frac{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}}{2^m}.$$

By using Theorem 4.1 and the same reasoning of Corollary 3.3 we obtain:

Theorem 4.2. *The analytic torsion of $\mathbb{R}\mathrm{P}^{2m}$ in relation to the representation*

$$\rho \simeq \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{p} \oplus \underbrace{\sigma_2 \oplus \cdots \oplus \sigma_2}_{N-p},$$

where σ_1 and σ_2 are respectively the trivial and non-trivial irreducible representations, is given by:

$$T_a(\mathbb{R}\mathrm{P}^{2m}, \rho) = T_a(\mathbb{R}\mathrm{P}^{2m})^{2p-N} \cdot T_a(S^{2m})^{N-p}.$$

5 Reidemeister torsion and Cheeger-Müller Theorem

In this section we study Reidemeister torsion on even dimensional real projective spaces. For aesthetic purposes we chose to work exclusively with real representations of $\pi_1(\mathbb{R}\mathrm{P}^{2m})$, but all the discussion made in this section is also valid for

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unitary representations of this group, with the same reasoning. Since even dimensional real projective spaces are non-orientable we use their odd-dimensional versions to obtain the important data required to develop our study. Throughout the section n will denote an arbitrary element of $\mathbb{N}\backslash\{1\}$ and we will specialize to the odd or even dimensional cases when necessary. Moreover, we will follow the treatment and notation used by Müller on [15] whenever it is convenient and in particular, given a vector space E of dimension r, we will use the notations $\det E := \Lambda^r(E)$ and $E^{-1} := E^*$, with the convention that $\det\{0\} \simeq \mathbb{R}$.

Let

$$C : 0 \to C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} \dots \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \to 0$$

be a chain complex of finite dimensional real vector spaces and denote by $Z_q(C)$ the kernel of ∂_q, by $B_q(C)$ the image of ∂_{q+1} and by $H_q(C)$ the quotient $Z_q(C)/B_q(C)$. We define the determinant line of the complex C, $\det C$:

$$\det(C) := \otimes_{q=0}^{n} (\det C_q)^{(-1)^q},$$

and also the set

$$\det H_*(C) := \otimes_{q=0}^{n} (\det H_q(C))^{(-1)^q}.$$

For each $q \in \{1, \dots, n\}$, let $b_q := \dim B_{q-1}(C)$ and choose elements $\theta_q \in \Lambda^{b_q}(C_q)$ such that $\partial_q\theta_q \neq 0$. By convention, with the intention of simplifying expression 1.3 of [15], we also set $\theta_0 := 1 =: \partial\theta_{n+1}$. For each $q \in \{0, \dots, n\}$, let $h_q := \dim H_q(C)$ and choose elements $0 \neq \mu_q \in \det H_q(C)$ and $\nu_q \in \Lambda^{h_q}(\ker \partial_q)$ such that $\pi(\nu_q) = \mu_q$, where $\pi : Z_q(V) \to H_q(V)$ is the canonical projection.

Within this framework, $\partial\theta_{q+1} \wedge \theta_q \wedge i(\nu_q)$ is a nonzero element of $\det C_q$, and by setting $\mu = \otimes_{q=0}^{n} (\mu_q)^{(-1)^{q+1}}$, where μ_q^{-1} denotes the element of $\det H_q(C)^{-1}$ such that $\mu_q^{-1}(\mu_q) = 1$, we may define the torsion $T(C)$ of the complex C, $T(C) \in (\det C) \otimes (\det H_*(C))^{-1}$, by the expression:

$$T(C) = \left(\bigotimes_{q=0}^{n} (\partial\theta_{q+1} \wedge \theta_q \wedge i(\nu_q))^{(-1)^q} \right) \otimes \mu,$$

which does not depend on the choices of θ_q, μ_q and ν_q. Note that, to simplify the expression (1.3) of [15], we adopted the convention $\theta_0 := \partial\theta_{n+1} := 1$.

The choice of specific elements $\omega \in \det C$ and $\mu \in \det H_*(C)$ establishes an isomorphism $\mathbb{R} \simeq (\det C) \otimes (\det H_*(C))^{-1}$ which sends $\lambda \in \mathbb{R}$ into $\lambda\omega\otimes\mu^{-1}$. This isomorphism allows us to consider $T(C)$ as a real number, denoted by $T(C, \omega, \mu)$, and therefore, it allows us to define the Reidemeister torsion $\tau(C, \omega, \mu)$ of the complex C in relation to the volumes ω and μ by:

$$\tau(C, \omega, \mu) := |T(C, \omega, \mu)|.$$

In this text we will consider the case where the chain complex C is obtained in the following way: we select the cell decomposition of $\mathbb{RP}^n$ formed by one cell of each dimension $\{e^0, \dots, e^n\}$ and we use this decomposition to induce a cell decomposition for S^n, which is then formed by two cells of each dimension $\{e_1^0, e_2^0, \dots, e_1^n, e_2^n\}$. We will consider $\mathbb{RP}^n$ as being embedded in S^n as a fundamental domain, so that the q-cells of S^n are translates of the q-cell of $\mathbb{RP}^n$ under the action of the fundamental group $\pi_1 := \pi_1(\mathbb{RP}^n)$. Note that under this procedure the real chain groups $C_q(S^n)$ generated by the q-cells of S^n become

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one-dimensional modules over the real group algebra $\mathbb{R}(\pi_1)$. We will assume without loss of generality that the embedding of $\mathbb{R}\mathrm{P}^n$ into S^n identifies the cells $e^0 \leftrightarrow e_1^0, \dots, e^n \leftrightarrow e_1^n$.

By fixing a representation

$$\rho \simeq \underbrace{\sigma_1 \oplus \cdots \oplus \sigma_1}_{p} \oplus \underbrace{\sigma_2 \oplus \cdots \oplus \sigma_2}_{N-p} : \pi_1(\mathbb{R}\mathrm{P}^n) \to \mathrm{GL}(\mathbb{R}^N),$$

for σ_1, σ_2 as defined on Corollary 3.3, we turn $\mathbb{R}^N$ into an $\mathbb{R}(\pi_1)$-module, and we become able to define the vector spaces:

$$C_q(\mathbb{R}\mathrm{P}^n, \rho) := C_q(S^n) \otimes_{\mathbb{R}(\pi_1)} \mathbb{R}^N, \quad q \in \{0, \dots, n\}$$

To simplify notation, specially on the reasoning to be developed below, we shall use (α, v) to represent the element $\alpha \otimes_{\mathbb{R}(\pi_1)} v \in C_q(\mathbb{R}\mathrm{P}^n, \rho)$. Since the boundary operator $\tilde{\partial}_q : C_q(S^n) \to C_{q-1}(S^n)$ originate an operator $\partial_{q,\rho} : C_q(\mathbb{R}\mathrm{P}^n, \rho) \to C_{q-1}(\mathbb{R}\mathrm{P}^n, \rho)$, by sending $(\alpha, v) \mapsto (\tilde{\partial}_q \alpha, v)$, our definition of $C_q(\mathbb{R}\mathrm{P}^n, \rho)$ produces a natural chain complex

$$C_\bullet(\mathbb{R}\mathrm{P}^n, \rho) : 0 \to C_n(\mathbb{R}\mathrm{P}^n, \rho) \overset{\partial_{n,\rho}}{\to} \dots \overset{\partial_{1,\rho}}{\to} C_0(\mathbb{R}\mathrm{P}^n, \rho) \to 0,$$

whose homology will be denoted by $H_\bullet(\mathbb{R}\mathrm{P}^n, \rho)$.

Let $\{u_1, \dots, u_N\}$ denote the canonical base of $\mathbb{R}^N$. The embedding of $\mathbb{R}\mathrm{P}^n$ into S^n and the cell decomposition fixed previously induce natural preferred bases $\omega_q = \{(e_1^q, u_i)\}_{i \in \mathbb{N}_N}$ for each vector space $C_q(\mathbb{R}\mathrm{P}^n, \rho)$.

We will also use $\{u_1, \dots, u_N\}$ and the cell decomposition of $\mathbb{R}\mathrm{P}^n$ to construct bases μ_q for the spaces $H_q(\mathbb{R}\mathrm{P}^n, \rho)$, but we need more concepts and reasoning to accomplish that. Our intention is to use the same method presented on the first paragraph of [15, page 729] to construct these bases, i. e., we will first fix orthonormal bases for the sets $\ker(\Delta^q_{\mathbb{R}\mathrm{P}^n,\rho})$, $q \in \{0, \dots, n\}$, then use de Rham isomorphism and Poincaré duality on these bases to obtain the desired bases for $H_q(\mathbb{R}\mathrm{P}^n, \rho)$. Note that since Poincaré duality does not hold for non-orientable manifolds, on the specific part where we apply this duality we will have to restrict our study to the cases where $n = 2m - 1$, but the results obtained in these cases will allow us to propose bases for $H_q(\mathbb{R}\mathrm{P}^{2m}, \rho)$ which will result in obtaining the same value for Reidemeister torsion that we obtained for analytic torsion on Theorem 4.2. It is important to mention that the Reidemeister torsion depends on the specific choice of the bases μ_q, so a different procedure to the one used on this text could lead to a different value of the torsion.

We start our construction observing that the elements of $\ker(\Delta^q_{\mathbb{R}\mathrm{P}^n,\rho})$ correspond via pullback by the covering projection $\pi : S^n \to \mathbb{R}\mathrm{P}^n$ to the elements of $\ker(\Delta^q_{S^n,N})$ which are equivariant in relation to ρ, where $\Delta^q_{S^n,N}$ denotes the Laplacian on $C^\infty(\Lambda^q S^n \otimes \mathbb{R}^N)$.

On [11, page 183], by using spherical coordinates $(\theta_k)_{k \in \{0,\dots,n\}}$ and representing the canonical metric of S^n by g, the authors describe orthonormal bases for $\ker(\Delta^q_{S^n})$, which are given by:

$$a_0 = \left\{ \frac{1}{\sqrt{\mathrm{vol}(S^n)}} \right\}, \quad a_n = \left\{ \frac{1}{\sqrt{\mathrm{vol}(S^n)}} \sqrt{|g|} d\theta_n \wedge \cdots \wedge d\theta_1 \right\}. \tag{14}$$

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By taking the tensor product of the elements of a_0 and a_n with the elements of the canonical base of $\mathbb{R}^N$ we obtain orthonormal bases $\tilde{a}_0$ and $\tilde{a}_n$ for $\ker(\Delta^q_{S^n,N})$:

$$\tilde{a}_0 = \left\{ \frac{1}{\sqrt{\mathrm{vol}(S^n)}} \otimes u_i \right\}_{i\in\mathbb{N}_N}, \ \tilde{a}_n = \left\{ \frac{1}{\sqrt{\mathrm{vol}(S^n)}} \sqrt{|g|} d\theta_n \wedge \cdots \wedge d\theta_1 \otimes u_i \right\}_{i\in\mathbb{N}_N}. \quad (15)$$

Using the canonical metric for $\mathbb{R}\mathrm{P}^n$, a form $\sum_{i=1}^{N} f_i \frac{1}{\sqrt{\mathrm{vol}(S^n)}} \otimes u_i$ is equivariant in relation to ρ if, and only if, $f_i = 0$ for $i \in \{p+1, \ldots, N\}$, and a form $\sum_{i=1}^{N} f_i \frac{1}{\sqrt{\mathrm{vol}(S^n)}} \sqrt{|g|} d\theta_n \wedge \cdots \wedge d\theta_1 \otimes u_i$ is equivariant if, and only if, n is odd and $f_i = 0$ for $i \in \{p+1, \ldots, n\}$; or n is even and $f_i = 0$ for $i \in \mathbb{N}_p$. Therefore, after normalization, we obtain the following orthonormal bases for $\ker(\Delta^q_{\mathbb{R}\mathrm{P}^n}, \rho)$:

$$\begin{aligned} b^n_0 &= \left\{ \frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^n)}} \otimes \overline{u_i} \right\}_{i\in\mathbb{N}_p}, \ b^{2m-1}_{2m-1} = \left\{ \frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m-1})}} \overline{\sqrt{|g|} d\theta_{2m-1} \wedge \cdots \wedge d\theta_1 \otimes u_i} \right\}_{i\in\mathbb{N}_p} \\ b^{2m}_{2m} &= \left\{ \frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}} \overline{\sqrt{|g|} d\theta_{2m} \wedge \cdots \wedge d\theta_1 \otimes u_i} \right\}_{i\in\{p+1,\ldots,N\}}, \end{aligned}$$

where the bar indicates we are using the projection of S^n onto $\mathbb{R}\mathrm{P}^n$ to descend the forms $\sqrt{|g|} d\theta_n \wedge \cdots \wedge d\theta_1 \otimes u_i$ and $1 \otimes u_i$ which are equivariant.

In the case $n = 2m-1$, we combine de Rham Isomorphism and Poincaré duality to form operators A_q, which when applied to the bases b^{2m-1}_q described above give rise to bases for $H_q(C(\mathbb{R}\mathrm{P}^{2m-1}, \rho))$. We have:

$$\begin{aligned} & A_{2m-1}\left(\frac{1}{\sqrt{\mathbb{R}\mathrm{P}^{2m-1})}} \overline{\sqrt{|g|} d\theta_{2m-1} \wedge \cdots \wedge d\theta_1 \otimes u_i} \right) \\ = & \int_{e^{2m-1}} \frac{1}{\sqrt{\mathbb{R}\mathrm{P}^{2m-1})}} \overline{\sqrt{|g|} d\theta_{2m-1} \wedge \cdots \wedge d\theta_1} (e^0_1, u_i) = \sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m-1})} (e^0_1, u_i), \\ & A_0 \left(\frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m-1})}} \otimes \overline{u_i} \right) = \frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m-1})}} \int_{e^0} 1 (e^{2m-1}_1, u_i) \\ = & \frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m-1})}} (e^{2m-1}_1, u_i) \end{aligned}$$

and the bases of $H_0(C(\mathbb{R}\mathrm{P}^{2m-1}, \rho))$ and $H_{2m-1}(C(\mathbb{R}\mathrm{P}^{2m-1}, \rho))$ are, respectively:

$$h^{2m-1}_0 = \left\{ \sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m-1})} (e^0_1, u_i) \right\}_{i\in\mathbb{N}_p}, \quad h^{2m-1}_{2m-1} = \left\{ \frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m-1})}} (e^{2m-1}_1, u_i) \right\}_{i\in\mathbb{N}_p}.$$

In the case $n = 2m$ we do not have a systematic method for choosing bases for $H_0(C(\mathbb{R}\mathrm{P}^{2m}, \rho))$ and $H_{2m}(C(\mathbb{R}\mathrm{P}^{2m}, \rho))$, so we use the odd dimensional case to choose, respectively, the following bases:

$$h^{2m}_0 = \left\{ \sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})} (e^0_1, u_i) \right\}_{i\in\mathbb{N}_p}, \quad h^{2m}_{2m} = \left\{ \frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}} (e^{2m}_1, u_i) \right\}_{i\in\{p+1,\ldots,N\}}.$$

Now that we have bases for $C_q(\mathbb{R}\mathrm{P}^n, \rho)$ and $H_q(C(\mathbb{R}\mathrm{P}^{2m}, \rho))$ we just need to understand the behavior of the operators $\partial_{q,\rho}$ to calculate the Reidemeister torsion of $\mathbb{R}\mathrm{P}^{2m}$.

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Since the boundary operators $\tilde{\partial}_{q+1} : C_{q+1}(S^n) \to C_q(S^n)$ satisfy

$$\tilde{\partial}_{q+1}(e_1^{q+1}) = \pm(e_1^q + (-1)^{q+1} e_2^q),$$

the operator $\partial_{q,\rho}$ acts on the base $\{(e_1^q, u_i)\}_{i\in\mathbb{N}_N}$ of $C_q(\mathbb{R}\mathrm{P}^n, \rho)$ by sending:

$$\begin{array}{llll}
(e_1^{2k}, u_i) & \mapsto & (e_1^{2k-1}, u_i) + (e_2^{2k-1}, u_i) = 2(e_1^{2k-1}, u_i); & 1 \leqslant i \leqslant p \\
(e_1^{2k}, u_i) & \mapsto & (e_1^{2k-1}, u_i) + (e_2^{2k-1}, u_i) = 0; & p+1 \leqslant i \leqslant N \\
(e_1^{2k-1}, u_i) & \mapsto & (e_1^{2k-2}, u_i) - (e_2^{2k-2}, u_i) = 0; & 1 \leqslant i \leqslant p \\
(e_1^{2k-1}, u_i) & \mapsto & (e_1^{2k-2}, u_i) - (e_2^{2k-2}, u_i) = 2(e_1^{2k-2}, u_i); & p+1 \leqslant i \leqslant N
\end{array}$$

and we obtain natural bases $\partial_{q+1}(b_{q+1}) \cup h_q \cup b_q$ for $C_q(\mathbb{R}\mathrm{P}^{2m})$, which are given by:

$$\begin{array}{ll}
\left\{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}(e_1^0, u_i)\right\}_{i\in\mathbb{N}_p} \bigcup \{2(e_1^0, u_i)\}_{i\in\{p+1,\ldots,N\}}, & \text{if } q = 0 \\
\{2(e_1^q, u_i)\}_{i\in\{p+1,\ldots,n\}} \bigcup \{(e_1^q, u_i)\}_{i\in\mathbb{N}_p}, & \text{if } q \text{ is even and } 1 \leqslant q < 2m \\
\{2(e_1^q, u_i)\}_{i\in\mathbb{N}_p} \bigcup \{(e_1^q, u_i)\}_{i\in\{p+1,\ldots,N\}}, & \text{if } q \text{ is odd and } 1 \leqslant q < 2m \\
\left\{\frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}}(e_1^{2m}, u_i)\right\}_{i\in\{p+1,\ldots,N\}} \bigcup \{(e_1^{2m}, u_i)\}_{i\in\mathbb{N}_p}, & \text{if } q = 2m.
\end{array}$$

By defining $\omega = \otimes_{q=0}^{2m} \omega_q$ we have:

$$\begin{aligned}
& \left(\bigotimes_{q=0}^{2m} (\partial\theta_{q+1} \wedge \theta_q \wedge i(\nu_q))^{(-1)^q}\right) \\
= & \left(\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}\right)^p 2^{N-p}\omega_0 \otimes \left(\bigotimes_{q=1}^{m-1} (2^p\omega_{2q-1})^{-1} \otimes (2^{N-p}\omega_{2q})\right) \\
& \otimes (2^p\omega_{2m-1})^{-1} \otimes \left(\frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}}\right)^{N-p} \omega_{2m},
\end{aligned}$$

so that:

$$\begin{aligned}
\tau_R(\mathbb{R}\mathrm{P}^{2m}, \rho) &= \left(\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}\right)^p 2^{N-p} \cdot \prod_{q=1}^{m-1} \left[(2^{N-p})(2^p)^{-1}\right] (2^p)^{-1} \\
&\times \left(\left(\frac{1}{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}}\right)^{N-p}\right) = \frac{2^{mN}}{2^{2mp}} \left(\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}\right)^{2p-N}.
\end{aligned} \tag{16}$$

Based on this value we can establish the validity of Cheeger-Müller Theorem for even dimensional real projective spaces, as stated in the next Theorem:

Theorem 5.1. *Let $m \in \mathbb{N}$ and let ρ be a representation of the fundamental group of $\mathbb{R}\mathrm{P}^{2m}$. The analytic torsion of the twisted de Rham complex $T_a(\mathbb{R}\mathrm{P}^{2m}, \rho)$ and the Reidemeister torsion $\tau_R(\mathbb{R}\mathrm{P}^{2m}, \rho)$ of $\mathbb{R}\mathrm{P}^{2m}$ in relation to ρ have the same value.*

Proof. By Theorem 4.1 of the present text and Theorem 2.3 of [19] we have:

$$T_a(\mathbb{R}\mathrm{P}^{2m}) = \frac{\sqrt{\mathrm{vol}(\mathbb{R}\mathrm{P}^{2m})}}{2^m}, \quad T_a(S^{2m}) = 1$$

so, from Theorem 4.2 and (16) it follows that

$$T_a(\mathbb{R}\mathrm{P}^{2m}, \rho) = \tau_R(\mathbb{R}\mathrm{P}^{2m}, \rho).$$

□

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Conflict of interest statement

The author Rafael Ferreira da Silva certify that he has NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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