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# NUMERICAL FOUNDATIONS OF ROOTFINDING ALGORITHMS BASED ON NEWTON-RAPHSON FOR NON-HOLOMORPHIC COMPLEX SYSTEMS USING WIRTINGER OPERATORS

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Abstract. Many nonlinear systems in $\mathbb{C}^n$, such as the power flow equations of electrical networks, are not holomorphic, and are therefore usually solved by Newton-Raphson after splitting every quantity into real and imaginary parts. In contrast to this approach, we develop in this paper the convergence theory of the Newton-Raphson-Wirtinger (NRW) method natively in $\mathbb{C}^n$. The Wirtinger operators $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ are derived as the unique coefficients of the widely-linear first-order approximation of $f$, which is equivalent to real Fréchet differentiability but is posed and computed without realification, and their Lipschitz continuity is shown to be equivalent to real Fréchet differentiability with a Lipschitz derivative. On this basis, we prove a Newton-Kantorovich semilocal convergence theorem and a local quadratic convergence theorem, which offer dufficient but not necessary conditions for convergence. These conditions are expressed by means of constants expressed in terms of the Wirtinger operators, yielding an a priori convergence certificate computable in complex arithmetic. We give a CBLAS/LAPACKE implementation based on a Schur reduction, with an operation count and an honest comparison against the realified method. Two applications illustrate the results. In a single-generator infinite-bus power flow problem, the Kantorovich constants are computed explicitly, the convergence certificate holds, and the iterates are verified against realified Newton-Raphson. In vortex states of the discretized Gross-Pitaevskii equation, the global phase invariance is removed by a structure-preserving phase pinning, Newton converges quadratically. We also prove an affine-covariant version of the Kantorovich theorem, invariant under all invertible widely-linear transformations of the equations, whose constant can be smaller than the product of the classical ones by orders of magnitude, as the vortex example indicates.

## 1. Introduction

Nonlinear algebraic systems in $\mathbb{C}^n$ arise routinely in many areas of natural sciences, such as Electrical Power Systems [14, 15, 21], Signal Processing and communications [18, 26], complex-valued Neural Networks [31], Control Theory [12], Electromagnetics [2, 35], Quantum Mechanics [19, 38] and many other fields. The problem is generally proposed as a rootfinding problem in $n$ complex dimensions

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$$f(z) = 0, \;\; f : \mathbb{C}^n \to \mathbb{C}^n. \tag{1}$$

However, the vast majority of the universe of practical complex functions fail the criterion of holomorphicity around the root, meaning that the traditional Newton-Raphson (NR) method is inapplicable because its first-order approximation machinery is unavailable in the complex domain. Generally, this is a consequence of the mathematical formulation of the particular problem in question that employs non-holomorphic functions such as the complex absolute value or the conjugation operation.

This burden is generally overcome by the **realification** of the problem – separating the function and variable into real and imaginary parts, as in $f = u + jv$ and $z = x + jy$, thus generating an equivalent realified problem

$$\begin{bmatrix} u\,(x,y) \\ \\ v\,(x,y) \end{bmatrix} = 0, \;\; u, v : \mathbb{R}^{2n} \to \mathbb{R}^{2n}. \tag{2}$$

and then applying the NR method on the realified problem, which is generally possible because while $f$ is not **complex-differentiable**, it generally is **real-differentiable**, that is, $u_x, u_y, v_x, v_y$ exist. Indeed, taking for instance the absolute value and complex conjugate, while not complex-differentiable they are real-differentiable in the entire plane.

Realification however demands analytical work to separate all quantities into real and imaginary parts, work that is unnecessary when the problem is solved directly in the complex domain. Furthermore, many problems in power systems, electromagnetics, and signal processing yield Wirtinger operators with compact complex structure (diagonal, sparse, or Hermitian) that realification obscures. Working natively in the complex domain keeps those structures visible, so that the equivalent Jacobian in complex domain maintains its sparsity structures that can be exploited in the solver algorithm, although for dense operators realification remains the cheaper option (Table 1).

The reason the original problem (1) is not directly solvable by classical NR while the realified problem (2) is comes from the fact that $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, despite being isomorphic as real vector spaces, are not isomorphic as algebras. Complex differentiation imposes the Cauchy-Riemann conditions, while the real derivative requires no such additional structure.

The Wirtinger calculus [42], developed for signal processing and estimation [5, 17, 25, 26, 32, 41], offers an alternative formulation for solving the rootfinding problem directly in complex domain. Wirtinger operators are already part of some applications literature, such as the family of gradient-descent methods—the *Wirtinger flow* algorithms [6–8, 43, 44] — designed for phase retrieval: they minimize a real-valued loss by gradient steps rather than solving a complex rootfinding system via a Newton step, and are distinct from the method developed here.

The Wirtinger operators provide a notion of first-order approximation for complex functions even when the traditional complex derivative does not exist. While solutions based on the NR method with these operators have been proposed [37], the convergence theory – local and semilocal – for such Wirtinger-enhanced NR methods had not been established in the same principled form as the classical Kantorovich framework [22, 23]. In general, the Wirtinger operators are presented as consequences of the existence of the real partial derivatives $\partial f/\partial x_k$ and $\partial f/\partial y_k$, that is, once $f$ is real-differentiable, then its Wirtinger operators exist and extend real-linearity, and the classical local and semilocal convergence theory is applied using the fact that $\mathbb{C}$ is a vector space over $\mathbb{R}$. This derivation frames the operators as a notational convenience for real calculus rather than as the primary complex structure — and, as a result, convergence theory for Wirtinger-based methods has been developed by treating $\mathbb{C}^n \cong \mathbb{R}^{2n}$, rather than reasoning within $\mathbb{C}^n$ as a real-linear algebra in its own right.

The main mechanism by which Wirtinger operators approximate complex functions is relaxing the requirement of a complex-linear Fréchet to requiring only real-linearity, thus substituting the

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differential operator for a real-linear map of the form $T(h) = Ah + B\overline{h}$, with $T : \mathbb{C}^n \to \mathbb{C}^m$, $A, B \in \mathbb{C}^{m\times n}$. In the signal processing literature, such maps are called *widely-linear*, a term introduced by Picinbono and Chevalier [32] for estimators that act jointly on an observation and its conjugate, in contrast with *strictly linear*, i.e. complex-linear, ones [1, 26, 36]. From the mathematics literature, every widely-linear map is real-linear, and conversely every real-linear map splits uniquely into a complex-linear and an antilinear part and is therefore widely-linear (Lemma 2). Therefore, both literatures are somewhat complementary in that they deal with the same objects but from different applications and points of view.

While the complex-linear and antilinear splitting of complex operators is classical in the analysis and linear algebra literature [33], the identification of widely-linear maps on $\mathbb{C}^n$ with real-linear maps on $\mathbb{R}^{2n}$ is not standard in that literature [1, 36]. We prove in this paper that for every widely-linear complex operator there exists a bijectively related real-linear mapping: the two names describe the same class of maps. At the same time, in the Wirtinger calculus literature [5, 17, 25, 41], real-differentiability is assumed first, and the question of whether the form $Ah + B\overline{h}$ exhausts all candidate approximations never arises because the equivalence serves estimation rather than differentiation.

In this paper we explore the bijection between widely-linear and real-linear maps and we show this bijection yields theoretical and practical results in both literatures. We first show that the widely-linear representation is exhaustive (Lemma 2), which enables the first-order approximation $f(z_0 + h) = f(z_0)+Ah+B\overline{h}+o(|h|)$ to be posed directly in $\mathbb{C}^n$, as an equivalent but realification-free form of real Fréchet differentiability (Corollary 1). We then prove that it is the most general real-linear first-order approximation, and the Wirtinger operators are defined as its unique coefficients (Theorem 3). We retain the term *widely-linear*, rather than real-linear, because it names the parametrization by the pair $(A, B)$, which is the structure exploited throughout the paper.

This paper bridges the concept of wide linearity and real linearity for complex operators by reversing the order of construction: while, in general, the Wirtinger operators are defined from the real partial derivatives of a real-differentiable $f$, in this paper these operators $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ are derived as the unique coefficients of the widely-linear approximation of $f$, the real partial derivatives being consequences rather than prerequisites — and for general maps $f : \mathbb{C}^n \to \mathbb{C}^m$, the square case $m = n$ being needed only for rootfinding. The hypothesis is the same as real Fréchet differentiability (Corollary 1); what changes is that it is stated and verified in widely-linear coordinates, without realification. Lipschitz continuity of those operators is then shown to be equivalent to real Fréchet differentiability with a Lipschitz derivative, with a uniform quadratic remainder (Theorem 4), establishing the bridge to classical analysis and enabling the Kantorovich convergence machinery to be applied directly in $\mathbb{C}^n$.

The main contributions of this paper therefore span several fronts:

(1) We state and prove the bijection between widely-linear and real-linear maps $\mathbb{C}^n \to \mathbb{C}^m$ (Lemma 2): every real-linear map admits a unique widely-linear representation $Ah + B\overline{h}$, and conversely. Although this identification is standard in the signal processing literature, it is not present in the Wirtinger calculus literature, where it has a specific consequence: the real-linear first-order approximation of real Fréchet calculus, ordinarily represented by means of realification as a $2m \times 2n$ real Jacobian, can be rewritten in widely-linear form, so that the first-order approximation of $f$ is posed natively in $\mathbb{C}^n$, without realification (Theorem 3, Corollary 1);

(2) We develop a Wirtinger-enhanced Newton-Raphson method, abbreviated as NRW (Newton-Raphson-Wirtinger). Instead of relying on real-differentiability to then prove well-definedness, existence and uniqueness of the Wirtinger operators, which is the traditional path in current literature, we adopt a $\mathbb{C}$-native approach: building on Lemma 2, the Wirtinger operators

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are defined as the unique coefficients of the most general real-linear first-order approximation, without first computing real partial derivatives or a realified Jacobian; we then show that this notion coincides with real Fréchet differentiability (Corollary 1) and that Lipschitz Wirtinger operators yield a uniform quadratic remainder (Theorem 4). This allows developing the NRW method not as a consequence of real-differentiability, but as a natural non-holomorphic generalization of the traditional NR method in the complex domain and although NRW and realified Newton-Raphson produce identical iterates (Remark 2), the NRW formulation provides convergence certificates and a numerical implementation that are stated and computed natively in $\mathbb{C}^n$;

(3) We then show a semilocal convergence theorem for the NRW method (Theorem 5), its affine-covariant version (Theorem 6), which is invariant under all invertible widely-linear transformations of the equations, including conjugation, and a local quadratic convergence theorem (Theorem 7), both stated and proved natively in $\mathbb{C}^n$, without invoking $\mathbb{C}^n \cong \mathbb{R}^{2n}$; the affine-covariant constant $\omega$ is a structural result whose lower bounds are computable along the Newton iterates (Section 4.2), though a closed-form computable upper bound remains an open problem;

(4) As practical consequences, we show that the Kantorovich constants $\beta$, $\eta$, $L$ can be expressed directly in terms of $\partial f/\partial z$ and $\partial f/\partial \overline{z}$, preserving the algebraic structure of the problem; we find a convergence certificate computable entirely from the complex expression for $f$: evaluate the Wirtinger operators of $f$ at the initial guess $z_0$, compute $h = \beta L \eta$, and verify $h \leq \frac{1}{2}$ – all in native complex arithmetic, with no expansion into real and imaginary parts.

(5) We present a CBLAS/LAPACKE implementation based on a Schur reduction of the augmented system, give its operation count, and identify when it is advantageous over realification: when either Wirtinger operator, $\partial f/\partial z$ or $\partial f/\partial \overline{z}$, is cheap to invert, since the reduction can pivot on either;

(6) We illustrate these practical results using a modern problem that benefits from these operators: power flow studies in Electrical Power Systems. We show a single-generator infinite-bus system for a simple example, illustrating the method, calculating the iterations and the Kantorovich constants. We then compare the workflow to the realified equivalent method, showing that Wirtinger operators allow for analytically simpler and more intuitive modeling of problems by enabling native complex-domain formulations;

(7) We apply the method to vortex states of the Gross-Pitaevskii equation, a large sparse problem in which the global phase invariance makes every root degenerate. We show how to pin the phase without destroying the structure of the Wirtinger operators (Proposition 8), and we use the example to show that the Kantorovich certificate is sufficient but not necessary: this particular problem does not suffice the certificate by orders of magnitude, yet the iteration converges quadratically, which motivates an affine-invariant version of the certificate.

## 2. Convergence theory of the NR method

In its most generalized form, the rootfinding problem is framed in abstract functional spaces [23]. Let $X$ and $Y$ be Banach spaces, and let $\Omega \subset X$ denote an open, non-empty subset and let $F : \Omega \to Y$ some non-linear function or mapping.

The objectives of the NR method and its convergence theory are: (i) finding minimum conditions under which a root $x^* \in \Omega$ of $F$ is guaranteed to exist; (ii) numerically approximating $x^*$ and finding the minimum restrictions to $F$ under which a numerical iterative approximation of $x^*$ is possible; and (iii) find the convergence rate of the iterations. To replicate the linearization step of the NR algorithm in this abstract setting, the derivative is substituted by the Fréchet derivative, capable of mapping changes between distinct normed spaces [9].

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**Definition 1** (Fréchet differentiability)**.** An operator $F : \Omega \to Y$ is said to be Fréchet differentiable at a point $x \in \Omega$ if there exists a bounded linear operator $A \in \mathcal{L}(X, Y)$ (this being the space of all linear operators between $X$ and $Y$) such that

$$\lim_{\|h\|_X \to 0} \frac{\|F(x+h) - F(x) - Ah\|_Y}{\|h\|_X} = 0 \tag{3}$$

If such an operator $A$ exists, it is unique and called the Fréchet derivative of $F$ at $x$, denoted as $F'(x)$ or $DF(x)$.

With a rigorous definition for the derivative established, we note that $DF(x)$ offers a linear approximation of $F$ at the particular $x$; formally, if $F$ is Fréchet at $x$, then there exists a residue map $r : \Omega \to Y$ such that

$$F(x+h) - F(x) = F'(x)h + r(h), \qquad \lim_{\|h\|_X \to 0} \frac{\|r(h)\|_Y}{\|h\|_X} = 0 \tag{4}$$

Therefore, for a first given $x_1$, one can find a $h$ that approximates the root of $F$ by setting $F(x_1 + h) = 0_Y$ and obtaining $h$ by considering the residue sufficiently small:

$$0_Y \approx F(x_1) + F'(x_1)\, h \tag{5}$$

and under the strict assumption that the linear operator $F'(x) \in \mathcal{L}(X, Y)$ possesses a bounded inverse in $\Omega$, one can find $h$ that satisfies (5), thus obtaining a second approximation $x_2 = x_1 + h$, and repeating this process by defining the generalized Newton-Raphson sequence $\{x_n\}_{n=0}^{\infty}$

$$x_{(n+1)} = x_n - [F'(x_n)]^{-1} F(x_n), \quad n = 0, 1, 2, \ldots \tag{6}$$

Analyzing the conditions under which the sequence $\{x_n\}$ exists, remains confined within $\Omega$, and converges to the solution $x^*$ constitutes a major pillar of numerical functional analysis [3, 29]. Convergence analysis is broadly divided into two frameworks: **semi-local convergence**, analyzing conditions for both the existence of the root $x^*$ and the convergence of the NR sequence using criteria evaluated exclusively at the initial guess $x_0$ [3, 23]; and **local convergence**, which assumes a priori that a true root $x^*$ exists and derives a radius around it within which any initial guess $x_0$ guarantees convergence [34].

2.1. **Semilocal convergence: Kantorovich's theorem.** The Newton-Kantorovich theorem serves as a landmark result for semi-local convergence, asserting that the behavioral characteristics of the operator evaluated at a single starting point are sufficient to dictate semi-local convergence properties [22].

**Theorem 1** (Newton-Kantorovich)**.** *Let $X$ and $Y$ be Banach spaces and $\Omega \subset X$ be an open convex set. Let $F : \Omega \to Y$ be Fréchet differentiable, and assume that for an initial point $x_0 \in \Omega$, the inverse operator $[F'(x_0)]^{-1}$ exists and belongs to $\mathcal{L}(Y, X)$. Suppose the following conditions are satisfied:*

1. ***(K1): Bounded inverse.*** $\left\|[F'(x_0)]^{-1}\right\|_{\mathcal{L}(Y,X)} \leq \beta$
2. ***(K2): First step magnitude.*** $\left\|[F'(x_0)]^{-1} F(x_0)\right\|_X \leq \eta$
3. ***(K3): Lipschitz continuity of the derivative.*** *The Fréchet derivative satisfies a Lipschitz condition in $\Omega$, meaning there exists a constant $L > 0$ such that for all $x, y \in \Omega$:*

$$\left\|F'(x) - F'(y)\right\|_{\mathcal{L}(X,Y)} \leq L \|x - y\|_X \tag{7}$$

*Define the Kantorovich constant $h$ as:*

$$h = \beta L \eta \tag{8}$$

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*If* $h \leq \frac{1}{2}$ *and the closed ball* $\overline{B}(x_0, r_1) \subset \Omega$*, where:*

$$r_1 = \frac{1 - \sqrt{1 - 2h}}{\beta L} \tag{9}$$

*then the sequence* $\{x_n\}$ *defined by the generalized Newton-Raphson iteration converges to a root* $x^*$ *of* $F(x) = 0$*, which is unique in the ball* $B(x_0, r_2) \cap \Omega$*, where:*

$$r_2 = \frac{1 + \sqrt{1 - 2h}}{\beta L} \tag{10}$$

The underlying analytical mechanism of the Newton-Kantorovich theorem relies on the *Majorant Principle*: rather than evaluating the multidimensional error trajectory natively within $Y$, the convergence dynamics of the Newton sequence are rigidly bounded by a scalar, real-valued quadratic polynomial $p : \mathbb{R} \to \mathbb{R}$, defined completely by the initial system bounds $\beta, \eta$, and $L$:

$$p(t) = \frac{1}{2}Lt^2 - \frac{1}{\beta}t + \frac{\eta}{\beta} \tag{11}$$

Let $\{t_k\}_{k=0}^{\infty}$ denote the scalar sequence generated by applying the classical one-dimensional Newton-Raphson method to $p(t)$ with an initial guess $t_0 = 0$:

$$t_{k+1} = t_k - \frac{p(t_k)}{p'(t_k)} \tag{12}$$

Under the canonical assumption that the Kantorovich constant satisfies $h = \beta L\eta \leq \frac{1}{2}$, the polynomial $p(t)$ possesses real roots, the smallest of which is isolated at:

$$t^* = \frac{1 - \sqrt{1 - 2h}}{\beta L} \tag{13}$$

The pivotal mechanism of Kantorovich's theorem is proving that the scalar sequence $\{t_k\}$ majorizes the multidimensional sequence $\{x_k\}$, by establishing a contractive mapping over the majorant tracking errors, it can be shown inductively that for all $k \geq 0$:

$$\left\|x_{(k+1)} - x_k\right\| \leq t_{k+1} - t_k, \quad \|x - x^*\| \leq t^* - t_k. \tag{14}$$

Consequently, the error of the complex system is bounded from above by the convergence rate of the scalar sequence $\{t_k\} \to t^*$. When $h < \frac{1}{2}$, the scalar derivative at the root is non-zero ($p'(t^*) \neq 0$), guaranteeing that $\{t_k\}$ converges quadratically. This yields the explicit error bound tracking expression for the system:

$$\|x_k - x^*\| \leq \frac{(2h)^{2^k - 1}}{2^{k-1}}\eta \tag{15}$$

Conversely, in the limiting singular configuration where $h = \frac{1}{2}$, the polynomial $p(t)$ possesses a double root at $t^* = \frac{1}{\beta L}$, forcing $p'(t^*) = 0$. In this distinct boundary scenario, the majorant scalar sequence drops to a linear convergence rate with a ratio of $\frac{1}{2}$, meaning the multidimensional system is only guaranteed to satisfy a linear convergence bound:

$$\|x_k - x^*\| \leq \frac{\eta}{2^{k-1}} \tag{16}$$

The constants $\beta$ and $L$ enter the theorem only through their product, and each depends on the norms chosen in $X$ and $Y$ and on how $F$ is scaled, whereas the Newton iterates themselves are invariant under affine transformations $F \mapsto MF$ of the problem. Affine-invariant formulations of the Kantorovich theorem [11] replace $\beta L$ by a single affine-covariant Lipschitz constant $\omega$, bounding $\left\|F'(x_0)^{-1}\left(F'(x) - F'(y)\right)\right\| \leq \omega \|x - y\|$, and generally yield less pessimistic convergence radii.

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Both forms are developed here for the NRW method: the classical one in Theorem 5, whose constants map most directly onto the Wirtinger operators, and the affine-covariant one in Theorem 6.

2.2. **Local and quadratic convergence results.** Kantorovich's theorem is seemingly sophisticated because it allows for little assumptions: the existence of the root is not given, and the initial state may not satisfy the necessary proximity restrictions. If these characteristics are given, that is, the root knowlingly exists and the initial guess guaranteedly converges to it, then the characteristically rapid rate of convergence of Newton's method is guaranteed [24, 30] in a simpler theorem.

**Theorem 2** (Quadratic convergence in Banach spaces)**.** *Let $F : \Omega \subset X \to Y$ Fréchet differentiable, where $X, Y$ are Banach spaces. Assume there exists a point $x^* \in \Omega$ such that $F(x^*) = 0$ and $[F'(x^*)]^{-1} \in \mathcal{L}(Y, X)$ exists. Furthermore, assume $F'$ is locally Lipschitz continuous in a neighborhood of $x^*$. Then, for any $x_0$ sufficiently close to $x^*$, the error sequence $e_n = x_n - x^*$ satisfies:*

$$\|e_{n+1}\|_X \leq C \|e_n\|_X^2 \tag{17}$$

*for a real positive $C$ independent of $n$.*

Thus local quadratic convergence in Banach spaces [29] is established. In practical terms, the quadratic convergence property means that the order of the error, when small enough, doubles at each iteration. In practicality, this means that given a good enough initial guess, the order of the error progresses so that the exponent of the error falls in a sequence such as $10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}$ thus reaching machine precision in just a few iterations. In other words, when the problem is well-posed – a root exists and the derivative on that root is not zero and is large enough – the method converges remarkably quickly, justifying the NR method's acclaim. An evaluation of this quadratic behavior against computational expenditure is historically balanced via Ostrowski's efficiency index [30], defined as $I = p^{1/m}$ where $p$ is the convergence order and $m$ represents operator evaluations.

## 3. Structural limitations vis-à-vis non-holomorphism

The algebraic elegance of the Kantorovich theorem rests upon the Banach Space framework that pivots around the critical assumption of the existence of the Fréchet derivative $F'(x)$. When the spaces are complex ($X = \mathbb{C}^n$, $Y = \mathbb{C}^m$) and $\mathcal{L}(X, Y)$ is understood as the space of complex-linear operators, Fréchet differentiability maps equivalently to the requirement of complex differentiability of every component of $F$ with respect to every variable. In other words, by requiring a complex-linear Fréchet derivative, the first-order approximation it yields is only available to holomorphic functions. We thus intuit that maybe we can rebuild the Fréchet differentiability for non-holomorphic functions by relaxing this requirement, that is, we seek an operator $L : \mathbb{C}^n \to \mathbb{C}^m$ that approximates $f$ but is only real-linear and not necessarily complex-linear. We first denote what exactly we mean by real-linearity.

**Definition 2** (Real-linear complex operator [25, 36])**.** A real-linear mapping $T : V \to W$, with $V$ and $W$ complex spaces, is such that $T(u + v) = T(u) + T(v)$ and $T(rv) = rT(v)$ for any $r \in \mathbb{R}$ and $u, v \in V$, that is, $T$ is linear when $V$ and $W$ are regarded as vector spaces over $\mathbb{R}$.

In some cases in the literature, real-linearity is defined in terms of linear combinations of the real and imaginary parts, i.e., $T : \mathbb{C} \to \mathbb{C}$ is real-linear if $T(a + jb) = aT(1) + bT(j)$ for any $a, b \in \mathbb{R}$ [33, Chapter 1] – a definition useful for algebra and geometrical studies. It is simple to prove that both definitions are equivalent.

At a first glance, this intuition does not require any extra steps. Theorem 1 is stated for general Banach spaces, and since $\mathbb{C}$ is a vector space over $\mathbb{R}$, the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ places complex rootfinding problems formally within its scope. One can always realify $f : \mathbb{C}^n \to \mathbb{C}^n$ into $F_R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, compute the $2n \times 2n$ real Jacobian $F_R'$, and apply the theorem with $\beta_R = \left\|[F_R'(x_0)]^{-1}\right\|$,

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$\eta_R = \left\|[F_R'(x_0)]^{-1}F_R(x_0)\right\|$, and $L_R$ the Lipschitz constant of $F_R'$. While formally correct, this route is analytically strenuous: it requires decomposing $f$ into real and imaginary parts, computing all four real partial derivatives, assembling the $2n \times 2n$ Jacobian, and then extracting $\beta_R$, $\eta_R$, $L_R$ from an operator that no longer exposes the complex structure of the original problem. The constants become increasingly opaque as $n$ grows, and the connection to the original function $f$ is obscured by the real expansion.

In other words, given a particular function $f$ which is, most of times, nonlinear, separating real and imaginary parts and calculating the Jacobian can prove to be cumbersome and analytically impractical, and depending on the nonlinearities simply impossible. This is also true for calculating the realified Jacobian. Moreover, the Kantorovich constants $\eta, L, \beta, h$ might also be difficult to find, making it harder to ascertain whether the initial condition chosen guarantees convergence.

We now show that we can develop the same method, and convergence theory, natively in $\mathbb{C}^n$ without resorting to $\mathbb{R}^{2n}$, that is invoking this isomorphism is theoretically unnecessary, with significant practical implications such as more easily finding the Kantorovich constants. We first show that there exists a pair of operators $\partial f/\partial z$ and $\partial f/\partial \overline{z}$, defined natively on $\mathbb{C}^n$, that carry exactly the first-order information of $f$. We show that these operators, named Wirtinger operators, are equivalent to real-differentiability; furthermore, we show that they yield a convergence theorem of the same form as Theorem 1, with Kantorovich constants expressible directly in terms of norms of those operators — quantities computable from the complex expression of $f$ without any real expansion.

3.1. **Restoring the first-order approximation.** We want to restore a notion of first-order approximation for non-holomorphic $f$. Throughout this section, consider $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$ with $\Omega$ open, where $m$ is not necessarily equal to $n$; the square case $m = n$ is only required later, when the approximation must be inverted (Remark 1). We write $z = x + jy \in \mathbb{C}^n$ with components $z_k = x_k + jy_k$, $k = 1, \ldots, n$, and $e_k$ for the $k$-th canonical basis vector of $\mathbb{C}^n$; conjugation of vectors and matrices is taken entrywise, and matrix-valued complex derivatives follow the conventions of [17]. The symbol $|\cdot|$ denotes the Euclidean norm on $\mathbb{C}^n$ or $\mathbb{C}^m$ (the modulus when the dimension is 1), and $\|\cdot\|$ the induced operator norm, both for matrices and for real-linear maps $T : \mathbb{C}^n \to \mathbb{C}^m$, that is, $\|T\| = \sup_{|h|=1} |T(h)|$.

As argued above, the first-order approximation we seek is a map that is real-linear but not necessarily complex-linear. The following lemma, a classical result of linear algebra and complex analysis [33, 36], characterizes all such maps: every real-linear map decomposes uniquely into a part proportional to $h$ and a part proportional to $\overline{h}$.

**Lemma 1** (Unique splitting of complex operators [33, 36])**.** *Any* real-linear *mapping $T : V \to W$ between complex vector spaces can be uniquely decomposed into a sum of a complex-linear part $P$ and an antilinear (conjugate-linear) part $Q$, that is, for any $c \in \mathbb{C}$ and $x, y \in V$,*

$$T = P + Q \begin{cases} P(cx + y) = cP(x) + P(y) \\ Q(cx + y) = \overline{c}Q(x) + Q(y) \end{cases} \tag{18}$$

*The decomposition is given explicitly by $P(h) = \frac{1}{2}[T(h) - jT(jh)]$ and $Q(h) = \frac{1}{2}[T(h) + jT(jh)]$. In particular, if $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$, there exist unique matrices $A, B \in \mathbb{C}^{m\times n}$ such that $T(h) = Ah + B\overline{h}$ for all $h \in \mathbb{C}^n$, given column-wise by $Ae_k = P(e_k)$ and $Be_k = Q(e_k)$.*

Following the literature discussed in Section 1, maps of the form $h \mapsto Ah + B\overline{h}$ are called **widely-linear**.

**Definition 3** (Widely-linear complex operator)**.** A widely-linear complex operator $T : \mathbb{C}^n \to \mathbb{C}^m$ is defined by two matrices $A, B \in \mathbb{C}^{m\times n}$ such that $T(h) = Ah + B\overline{h}$. Since $\left|\overline{h}\right| = |h|$, its operator norm satisfies $\|T\| \leq \|A\| + \|B\|$.

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**Lemma 2** (Widely-linear and real-linear maps)**.** *A map $T : \mathbb{C}^n \to \mathbb{C}^m$ is real-linear if and only if it is widely-linear. More precisely:*

*(i) every widely-linear map $T(h) = Ah + B\overline{h}$ is real-linear;*

*(ii) (exhaustiveness) every real-linear map $T$ is widely-linear, that is, there exist $A, B \in \mathbb{C}^{m\times n}$ with $T(h) = Ah + B\overline{h}$ for all $h \in \mathbb{C}^n$; and*

*(iii) (uniqueness) the pair $(A, B)$ in (ii) is unique.*

*Hence the correspondence $(A, B) \leftrightarrow T$ is a bijection between $\mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n}$ and the real-linear maps $\mathbb{C}^n \to \mathbb{C}^m$.*

*Proof.* (i) For $t \in \mathbb{R}$, $T(th) = A(th) + B\overline{(th)} = t\,T(h)$, and additivity is immediate. (ii) This is the existence part of Lemma 1. (iii) If $Ah + B\overline{h} = A'h + B'\overline{h}$ for all $h$, let $\Delta A = A - A'$ and $\Delta B = B - B'$; taking $h = e_k$ and $h = je_k$ gives $(\Delta A + \Delta B)e_k = 0$ and $j(\Delta A - \Delta B)e_k = 0$, hence $\Delta A e_k = \Delta B e_k = 0$ for every $k$, so $A = A'$ and $B = B'$. □

Lemma 2 motivates the ansatz. The first-order approximation we seek is real-linear, and a real-linear map between $\mathbb{C}^n$ and $\mathbb{C}^m$ is, in the real Fréchet setting of Definition 1, ordinarily represented by means of realification: $h$ and $f$ are split into real and imaginary parts and the approximation is written as a $2m \times 2n$ real Jacobian acting on $(\operatorname{Re} h, \operatorname{Im} h)$, with the drawbacks discussed at the beginning of this section. The bijection of Lemma 2 allows any real-linear approximation to be rewritten, uniquely, in the widely-linear form $Ah + B\overline{h}$, with $A$ and $B$ acting directly on $h$ and $\overline{h}$; the approximation can therefore be posed, and later computed, natively in $\mathbb{C}^n$, without realification. We apply this bijection to the real-linear realified Fréchet derivative and relate it to a widely-linear map, that is, we ask whether there exist matrices $A, B \in \mathbb{C}^{m\times n}$ such that, for $h \in \mathbb{C}^n$,

$$f(z_0 + h) - f(z_0) = Ah + B\overline{h} + r(h), \text{ where } r(h) = o(|h|) \Leftrightarrow \lim_{h\to 0} \frac{|r(h)|}{|h|} = 0. \tag{19}$$

The remainder $r(h) \in \mathbb{C}^m$ need not be quadratic; "small-$o$" means only that it tends to zero faster than $h$ itself. By Lemma 2, the ansatz (19) is exhaustive: no real-linear first-order approximation of $f$ lies outside the form $Ah + B\overline{h}$. It is this exhaustiveness that allows (19) to be posed directly in widely-linear form: rather than computing a real derivative in realified coordinates and rewriting it in terms of $h$ and $\overline{h}$, we ask for the coefficients $A$ and $B$ themselves. The hypothesis is not weaker than real-differentiability — by Corollary 1 the two are equivalent — but it is stated natively in $\mathbb{C}^n$.

We now answer two questions:

- The *downstream* question (from the approximation to the operators): when $A$ and $B$ satisfying (19) exist, are they unique, and how do they relate to the real partial derivatives $\partial f/\partial x_k$ and $\partial f/\partial y_k$; and
- The *upstream* question (from the operators to real-differentiability): under which conditions on $A$ and $B$, as functions of $z_0$, is the remainder of (19) controlled uniformly over a whole region, as the convergence theory requires, rather than only pointwise?

Theorem 3 answers the downstream question: if $A$ and $B$ satisfying (19) exist, they are unique; those unique coefficients are the Wirtinger operators.

**Theorem 3** (The widely-linear first-order approximation)**.** *Let $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$, $z_0 \in \Omega$, and suppose there exist $A, B \in \mathbb{C}^{m\times n}$ satisfying (19). Then:*

*(i) $A$ and $B$ are unique. They are called the* ***Wirtinger operators*** *of $f$ at $z_0$, denoted $A = \partial f/\partial z|_{z_0}$ and $B = \partial f/\partial\overline{z}|_{z_0}$, and the corresponding widely-linear approximation is*

$$L_{(f,z_0)}(h) := \left.\frac{\partial f}{\partial z}\right|_{z_0} h + \left.\frac{\partial f}{\partial \overline{z}}\right|_{z_0} \overline{h}, \tag{20}$$

*which depends on both $f$ and $z_0$. The $k$-th columns of the Wirtinger operators are denoted $\partial f/\partial z_k$ and $\partial f/\partial\overline{z}_k \in \mathbb{C}^m$; and*

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*(ii) The real partial derivatives $\partial f/\partial x_k$ and $\partial f/\partial y_k \in \mathbb{C}^m$, $k = 1, \ldots, n$, exist at $z_0$ and are recovered from the Wirtinger operators by*

$$\begin{cases} \dfrac{\partial f}{\partial x_k} = \dfrac{\partial f}{\partial z_k} + \dfrac{\partial f}{\partial \overline{z}_k} \\ \dfrac{\partial f}{\partial y_k} = j\left(\dfrac{\partial f}{\partial z_k} - \dfrac{\partial f}{\partial \overline{z}_k}\right) \end{cases}, \quad k = 1, \ldots, n. \tag{21}$$

*Proof.* (i) Suppose $(A_1, B_1)$ and $(A_2, B_2)$ both satisfy (19). Let $\Delta A = A_1 - A_2$ and $\Delta B = B_1 - B_2$; subtracting the two instances of (19) gives

$$\Delta A\, h + \Delta B\, \overline{h} = o\left(|h|\right). \tag{22}$$

Fix $k \in \{1, \ldots, n\}$. Probing along $h = te_k$, $t \in \mathbb{R}^+$, dividing by $t$ and letting $t \to 0$: $(\Delta A + \Delta B)\, e_k = 0$. Probing along $h = jte_k$ in the same way: $j\,(\Delta A - \Delta B)\, e_k = 0$. Together, $\Delta A e_k = \Delta B e_k = 0$; since $k$ is arbitrary, every column of $\Delta A$ and $\Delta B$ vanishes and $A$ and $B$ are unique. Since they depend on $f$ and $z_0$, they are denoted as in (20).

(ii) Restricting (19) to $h = te_k$, $t \in \mathbb{R}$, and dividing by $t$ shows that the limit defining $\partial f/\partial x_k$ at $z_0$ exists and equals $(A + B)\, e_k$; restricting to $h = jte_k$ and dividing by $t$ shows that $\partial f/\partial y_k$ exists at $z_0$ and equals $j\,(A - B)\, e_k$. These are exactly the relations (21). □

**Corollary 1** (Wirtinger and real Fréchet differentiability)**.** *Let $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$ and $z_0 \in \Omega$. Then $f$ is Wirtinger differentiable at $z_0$, that is, (19) holds for some $A, B \in \mathbb{C}^{m\times n}$, if and only if $f$ is real Fréchet differentiable at $z_0$, with $\mathbb{C}^n$ and $\mathbb{C}^m$ regarded as real normed spaces in Definition 1. In that case $Df\,(z_0) = L_{(f,z_0)}$.*

*Proof.* If (19) holds, the map $h \mapsto Ah + B\overline{h}$ is real-linear by Lemma 2 (i), so (19) is precisely Definition 1 and $Df\,(z_0) = L_{(f,z_0)}$. Conversely, if $f$ is real Fréchet differentiable at $z_0$, its derivative $Df\,(z_0)$ is real-linear, hence of the form $h \mapsto Ah + B\overline{h}$ by Lemma 2 (ii), and (19) holds; by Theorem 3 (i), $A$ and $B$ are the Wirtinger operators. □

In short, Theorem 3 shows that if $f$ admits a degree-1 polynomial approximation in $(h, \overline{h})$, the coefficients are unique and constitute the Wirtinger operators (20); the real partial derivatives $\partial f/\partial x_k$ and $\partial f/\partial y_k$ are then derived from (21) as a consequence, not prerequisites.

The Wirtinger operators therefore restore a notion of first-order approximation without presupposing complex-linearity. Real-linearity of the approximation is presupposed, since by Lemma 2 the maps $h \mapsto Ah + B\overline{h}$ are exactly the real-linear ones; what is not presupposed is a splitting of $f$ and $z$ into real and imaginary parts. Nor is a weaker hypothesis than real-differentiability being used: by Corollary 1, (19) is real Fréchet differentiability written in widely-linear form. What changes is the order of construction: the existence of these operators does not require the real partial derivatives to be computed beforehand; on the contrary, their existence at $z_0$ implies that all $2n$ real partial derivatives exist there.

Three properties of these operators are noteworthy: first, suppose $f$ is complex-differentiable; then by applying the Cauchy-Riemann equations to each component $f_i$ in each variable $z_k$, we see that the first operator coincides with the traditional complex Jacobian $[\partial f_i/\partial z_k]$ when it exists; thus Wirtinger operators extend the conventional complex derivative to the broader class of non-holomorphic functions.

Second property: by construction, $z$ and $\overline{z}$ are independent, that is, applying (19) to the identity map and to the conjugation map,

$$\frac{\partial \overline{z}}{\partial \overline{z}} = \frac{\partial z}{\partial z} = I_n \text{ and } \frac{\partial \overline{z}}{\partial z} = \frac{\partial z}{\partial \overline{z}} = 0_{n\times n}. \tag{23}$$

And third: the equality

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$$\frac{\partial f(z)}{\partial \overline{z}} = 0_{m\times n} \tag{24}$$

is equivalent to complex differentiability; writing $f_i = u_i + jv_i$ for the $i$-th component of $f$ and expanding each entry via $\partial f_i/\partial \overline{z}_k = \frac{1}{2}(\partial f_i/\partial x_k + j\,\partial f_i/\partial y_k)$ recovers exactly the Cauchy-Riemann equations, as verified below for $i = 1,\dots,m$ and $k = 1,\dots,n$.

$$\frac{\partial f_i}{\partial \overline{z}_k} = 0 \Leftrightarrow \frac{\partial u_i}{\partial x_k} + j\frac{\partial v_i}{\partial x_k} + j\left(\frac{\partial u_i}{\partial y_k} + j\frac{\partial v_i}{\partial y_k}\right) = 0 \Leftrightarrow \begin{cases} \dfrac{\partial u_i}{\partial x_k} - \dfrac{\partial v_i}{\partial y_k} = 0 \\ \dfrac{\partial u_i}{\partial y_k} + \dfrac{\partial v_i}{\partial x_k} = 0 \end{cases} \tag{25}$$

yielding the Cauchy-Riemann equations of every component of $f$ in every variable, which are themselves equivalent to complex differentiability. This equivalence also holds pointwise: if (24) holds at a specific point $z$ *and* $f$ *is real-differentiable there*, then $f$ is complex-differentiable at $z$, its real derivative $h \mapsto \partial f/\partial z|_z\, h$ being complex-linear. Naturally, if for a given $f$, (24) holds as an identity over all its domain, $f$ is differentiable all over its domain, thus holomorphic. If, moreover, $\partial f/\partial \overline{z}$ is continuous — as is the case under the hypotheses of Theorem 4 — and (24) holds everywhere except on a set with empty interior, such as a collection of isolated points, then by continuity it holds identically and $f$ is holomorphic. Informally, the intuition is that if $f$ does not depend on $\overline{z}$, then it behaves as a complex-differentiable function; this should be understood as a heuristic, not a definition.

Therefore the formal answer to the downstream question is: Theorem 3 proves that any degree-1 polynomial approximation in $(h, \overline{h})$ satisfying (19) is unique and its coefficients are the Wirtinger operators. Lemma 2 confirms that the approximation is real-linear, and the connection to partial derivatives follows from Part (ii) of the theorem.

We note that the existence of the Wirtinger operators is proven with no limit ever taken "in the direction $z$" or "in the direction $\overline{z}$". This circumvents the problem that for a non-holomorphic $f$ no complex-linear map $h \mapsto Ah$ alone satisfies $f(z_0 + h) - f(z_0) = Ah + o(|h|)$ — for $n = m = 1$, the difference quotient $(f(z_0 + h) - f(z_0))/h$ has no limit. The operators are defined as *coefficients* — the components of an existing degree-1 polynomial approximation — and the identification of their columns with the combinations $\frac{1}{2}(\partial f/\partial x_k \mp j\,\partial f/\partial y_k)$ is a computation, not a definition, which exists only once the partial derivatives are shown to exist in Part (ii). This is why the calculus applies to exactly the functions that defeat the complex derivative: it never asks of $f$ more than the existence of the approximation (19).

3.2. **Restoring real Fréchet differentiability.** We now answer the upstream question. Approximation (19) is a pointwise statement: the remainder is $o(|h|)$, with no information on how fast it vanishes nor on how this rate varies with $z_0$. The convergence theory, on the other hand, demands a remainder controlled quadratically and uniformly over a whole region, and we ask under which conditions on the Wirtinger operators this holds. For the remainder of the paper we denote, for brevity and for $p \in \Omega$,

$$H_f(p) \coloneqq \left.\frac{\partial f}{\partial z}\right|_{z=p}, \quad W_f(p) \coloneqq \left.\frac{\partial f}{\partial \overline{z}}\right|_{z=p}, \quad H_f(p), W_f(p) \in \mathbb{C}^{m\times n}. \tag{26}$$

**Theorem 4** (Lipschitz Wirtinger operators and the quadratic remainder)**.** *Let $U \subseteq \mathbb{C}^n$ be open and convex, and let $f : U \to \mathbb{C}^m$ have Wirtinger operators at every point of $U$ that are Lipschitz in $U$, that is, there exists $\omega > 0$ such that for all $p, q \in U$*

$$\|H_f(p) - H_f(q)\| \le \omega\,|p - q|, \quad \|W_f(p) - W_f(q)\| \le \omega\,|p - q|. \tag{27}$$

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*Then $f$ is real Fréchet differentiable at every $p \in U$ with derivative $L_{(f,p)}$, the map $p \mapsto L_{(f,p)}$ is Lipschitz in $U$ with constant $2\omega$, and for all $z_0 \in U$ and $h$ with $z_0 + h \in U$ the widely-linear approximation has quadratic remainder:*

$$\left|f(z_0+h) - f(z_0) - L_{(f,z_0)}(h)\right| \leq \omega |h|^2 . \tag{28}$$

*Proof.* . By Corollary 1, $f$ is real Fréchet differentiable at every $p \in U$ with $Df(p) = L_{(f,p)}$ with $L$ as in (20). By Definition 3 and (27),

$$\left\|L_{(f,p)} - L_{(f,q)}\right\| \leq \|H_f(p) - H_f(q)\| + \|W_f(p) - W_f(q)\| \leq 2\omega |p-q| . \tag{29}$$

Now fix $z_0$ and $h$ with $z_0 + h \in U$; by convexity, the segment $[z_0, z_0+h]$ lies in $U$. By the chain rule, the curve $g(t) = f(z_0 + th)$, $t \in [0,1]$, is differentiable with $g'(t) = L_{(f,z_0+th)}(h)$, which by (29) is Lipschitz, hence continuous, in $t$. Since $f$ is vector- and complex-valued, the mean value theorem does not yield equality at a single intermediate point; instead we apply the fundamental theorem of calculus to the real and imaginary parts of each component of $g$:

$$f(z_0+h) - f(z_0) - L_{(f,z_0)}(h) = \int_0^1 \left(L_{(f,z_0+th)} - L_{(f,z_0)}\right)(h)\, dt. \tag{30}$$

Bounding the integrand by (29),

$$\left|f(z_0+h) - f(z_0) - L_{(f,z_0)}(h)\right| \leq \int_0^1 2\omega |th|\,|h|\, dt = \omega |h|^2 . \tag{31}$$

□

Therefore, we have that if $H_f$ and $W_f$ are Lipschitz in $U$, the first-order approximation $L_{(f,z_0)}$ is not only a first-order approximation, but quadratic thus giving the (real) Fréchet derivative of $f$ at $z_0$ with a remainder bound that is uniform over $U$. That is, under Lipschitz-continuity of Wirtinger differentiability, the widely-linear approximation (20) is promoted from a simple linear approximation whose remainder has an order less than the disturbance, i.e., "small-$o$" notation $o(|h|)$ of (19), to a better approximation where the remainder is genuinely of quadratic order, that is, "big-$O$" notation $O(|h|^2)$.

3.3. **Fundamental theorem of calculus for Wirtinger operators.** We finally conclude that if a function has Lipschitz Wirtinger operators in some domain, the widely-linear approximation (20) at $z_0$ has a quadratic remainder. More importantly, however, we note corollary 2.

**Corollary 2** (The continuity equivalence)**.** *Let $U \subseteq \mathbb{C}^n$ be open and $f : U \to \mathbb{C}^m$. The following are equivalent:*

*(a) the real partial derivatives $\partial f/\partial x_k$ and $\partial f/\partial y_k$, $k = 1, \ldots, n$, exist and are Lipschitz in $U$;*

*(b) $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ exist and are Lipschitz in $U$;*

*(c) $f$ is real Fréchet differentiable at every point of $U$ and its real derivative $Df : U \to \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}^m)$ is Lipschitz in $U$;*

*(d) the widely-linear approximation $L_{(f,p)}$ exists at every $p \in U$ and $p \mapsto L_{(f,p)}$ is Lipschitz in $U$.*

*A function satisfying these conditions is called Lipschitz-Wirtinger-differentiable, or simply Lipschitz-Wirtinger.*

*Proof.* (a)⇒(c): Lipschitz partial derivatives are continuous, and by the classical result that continuous partial derivatives imply Fréchet differentiability, $f$ is real Fréchet differentiable in $U$ with $Df(p)\,h = \sum_{k=1}^{n} [\partial f/\partial x_k \operatorname{Re}(h_k) + \partial f/\partial y_k \operatorname{Im}(h_k)]$. If $\omega$ is a common Lipschitz constant of the partial derivatives, the triangle and Cauchy–Schwarz inequalities give $\|Df(p) - Df(q)\| \leq \sqrt{2n}\,\omega |p-q|$.
(c)⇒(d): By Corollary 1, $L_{(f,p)}$ exists at every $p \in U$ and $L_{(f,p)} = Df(p)$, so the Lipschitz property is the same.

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(d)⇒(b): By Lemma 1,

$$H_f(p)\,e_k = \tfrac{1}{2}\left[L_{(f,p)}(e_k) - jL_{(f,p)}(je_k)\right] \text{ and} \tag{32}$$

$$W_f(p)\,e_k = \tfrac{1}{2}\left[L_{(f,p)}(e_k) + jL_{(f,p)}(je_k)\right], \tag{33}$$

so each column of $H_f$ and $W_f$ varies by at most $\left\|L_{(f,p)} - L_{(f,q)}\right\|$; since the operator norm of an $m \times n$ matrix is at most $\sqrt{n}$ times its largest column norm, $H_f$ and $W_f$ are Lipschitz.
(b)⇒(a): By Theorem 3 (ii) the partial derivatives exist and, by (21), $\partial f/\partial x_k = (H_f + W_f)\,e_k$ and $\partial f/\partial y_k = j\,(H_f - W_f)\,e_k$ are bounded linear combinations of Lipschitz functions, hence Lipschitz. □

This is then used to prove the fundamental theorem of calculus for Wirtinger derivatives:

**Corollary 3** (Fundamental theorem of calculus for Wirtinger operators)**.** *For a Lipschitz-Wirtinger $f : U \to \mathbb{C}^m$ in some convex domain $U \subseteq \mathbb{C}^n$, and any segment $[z_0, z_0 + h] \subset U$,*

$$f(z_0 + h) - f(z_0) = \int_0^1 \left[\left.\frac{\partial f}{\partial z}\right|_{(z_0+th)} h + \left.\frac{\partial f}{\partial \overline{z}}\right|_{(z_0+th)} \overline{h}\right] dt. \tag{34}$$

*Proof.* . Under the hypothesis (c) of corollary 2, the curve $t \mapsto f(z_0 + th)$ is continuously differentiable in the real variable $t$ with derivative

$$\frac{d}{dt} f(z_0 + th) = \left.\frac{\partial f}{\partial z}\right|_{(z_0+th)} h + \left.\frac{\partial f}{\partial \overline{z}}\right|_{(z_0+th)} \overline{h}, \tag{35}$$

and the proposed integral form (34) follows from the fundamental theorem of calculus applied to the real and imaginary parts of each component, the integrand being continuous. □

Corollary 3 gives us a way to reconstruct $f$ from its Wirtinger operators, which is the equivalent of the fundamental theorem of calculus, and will be instrumental in proving the generalized version of the semilocal convergence for the new proposed method because it enables the induction process known as Kantorovich's iterations.

*Remark* 1 (Rectangular and square maps). All results in this section hold for $f : \mathbb{C}^n \to \mathbb{C}^m$ with $m$ not necessarily equal to $n$: the Wirtinger operators are $m \times n$ matrices and the widely-linear approximation is a real-linear map $\mathbb{C}^n \to \mathbb{C}^m$. For $n = m = 1$ the Wirtinger operators are scalars, the norms reduce to moduli and (21) reduces to the familiar $\partial/\partial x = \partial/\partial z + \partial/\partial \overline{z}$, $\partial/\partial y = j\,(\partial/\partial z - \partial/\partial \overline{z})$. Rootfinding, however, requires invertibility of the widely-linear approximation, which is only possible when the real dimensions of domain and codomain agree (Lemma 3); hence, from Section 4 onward, $f : \mathbb{C}^n \to \mathbb{C}^n$.

## 4. The Newton-Raphson-Wirtinger method

Consider the initial problem (1), where $f : \mathbb{C}^n \to \mathbb{C}^n$ (Remark 1); take some $z_1$ that approximates the sought root and write the widely-linear approximation at $z_1$:

$$f(z_1 + h_1) = f(z_1) + \left[\frac{\partial f}{\partial z}\right]_{z_1} h_1 + \left[\frac{\partial f}{\partial \overline{z}}\right]_{\overline{z_1}} \overline{h_1} + O\left(|h_1|^2\right) \tag{36}$$

thus, for $h_1$ appropriately small, an approximation for $f(z_1 + h_1) = 0$ is

$$0 = f(z_1) + \left[\frac{\partial f}{\partial z}\right]_{z_1} h_1 + \left[\frac{\partial f}{\partial \overline{z}}\right]_{\overline{z_1}} \overline{h_1} \tag{37}$$

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thus obtaining a new approximation $z_2 = z_1 + h_1$ of the root; this process then becomes the **Newton-Raphson-Wirtinger** iteration

$$z_{(k+1)} = h_k + z_k, \text{ where } 0 = f(z_k) + \left[\frac{\partial f}{\partial z}\right]_{z_k} h_k + \left[\frac{\partial f}{\partial \overline{z}}\right]_{\overline{z_k}} \overline{h_k}. \tag{38}$$

Now we want to prove that the sequence $\{z_k\}_{k=1}^{\infty}$ generalizes the same properties as the traditional NR method, that is: the NRW method has the same semi-local and local convergence properties as NR.

*Remark* 2 (Equivalence with realified Newton-Raphson). The NRW iterate sequence is identical to the one produced by applying classical Newton-Raphson to the realified system $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $F(x, y) = (\operatorname{Re} f(x + jy), \operatorname{Im} f(x + jy))$.

To see this, take the conjugate of the NRW step equation $Ah_k + B\overline{h_k} = -f_k$:

$$\overline{A}\,\overline{h_k} + \overline{B}\,h_k = -\overline{f_k}. \tag{39}$$

Stacking the original equation and its conjugate gives the $2n \times 2n$ augmented system

$$\begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix} \begin{bmatrix} h_k \\ \overline{h_k} \end{bmatrix} = \begin{bmatrix} -f_k \\ -\overline{f_k} \end{bmatrix}. \tag{40}$$

The left-hand matrix is precisely the realified Jacobian of $F$ expressed in the basis $\{z, \overline{z}\}$ rather than $\{x, y\}$: the two bases are related by the unitary change of variables $z = x + jy$, $\overline{z} = x - jy$, which is an isomorphism of $\mathbb{R}^{2n}$. Both formulations therefore solve the same real linear system at each step and produce the same sequence $\{z_k\}$.

The NRW formulation is preferred not because it yields different iterates, but because it keeps the problem in $\mathbb{C}^n$: the Wirtinger operators $A$ and $B$ are computed directly from the complex expression for $f$, whereas realification requires expanding every quantity into real and imaginary parts, obscuring the algebraic structure (e.g., diagonal or Hermitian Jacobians in power-flow problems).

Notably, the NRW method generalizes the NR method for complex functions: indeed, if $f(z)$ is holomorphic around its root, then $\partial f/\partial \overline{z} \equiv 0$, which as shown in (24) and (25) is equivalent to complex differentiability and $\partial f/\partial z$ is equal to the traditional derivative. Therefore, (38) becomes the traditional NR method. In other words, the traditional NR is a particular version of the NRW method when $f(z)$ is complex-differentiable.

4.1. **Semilocal convergence analysis.** We now turn to proving the equivalent version of Kantorovich's theorem for the new NRW method. We first prove that if a function is Wirtinger differentiable, then with a few extra hypotheses the corresponding widely-linear operator is invertible and bounded.

**Lemma 3** (Invertibility of the widely-linear derivative)**.** *Let $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be Wirtinger differentiable at $z_0$, with $A = [\partial f/\partial z]_{z_0}$, $B = [\partial f/\partial \overline{z}]_{z_0} \in \mathbb{C}^{n\times n}$, and $L_{(f,z_0)}(h) = Ah + B\overline{h}$. The necessary and sufficient condition for $L_{(f,z_0)}$ to be invertible is that the $2n \times 2n$ augmented matrix*

$$\mathcal{J} = \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix} \tag{41}$$

*be invertible. In particular, for $n = 1$ the condition reduces to $|A|^2 - |B|^2 \neq 0$). When $A$ is itself invertible and the Schur complement $S = A - B\overline{A}^{-1}\overline{B}$ is invertible, the inverse can be written explicitly as*

$$\left[L_{(f,z_0)}\right]^{-1}[w] = S^{-1}w - S^{-1}B\overline{A}^{-1}\,\overline{w}. \tag{42}$$

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*Alternatively, one could instead pivot on $B$: if $B$ is invertible and the reduced matrix $R = \overline{B} - \overline{A}B^{-1}A$ is invertible, then eliminating $\overline{h}$ from the first block row via $\overline{h} = B^{-1}(w - Ah)$ and substituting into the second gives $Rh = \overline{w} - \overline{A}B^{-1}w$, yielding*

$$\left[L_{(f,z_0)}\right]^{-1}[w] = R^{-1}\overline{w} - R^{-1}\overline{A}B^{-1}w. \tag{43}$$

*If neither $A$ nor $B$ is well-conditioned, one solves the full augmented system $\mathcal{J}[h;\overline{h}]^\intercal = [w;\overline{w}]^\intercal$ with $LU$ pivoting.*

*Proof.* . The augmented system $\mathcal{J}[h;\overline{h}]^\intercal = [w;\overline{w}]^\intercal$ is equivalent to $L_{(f,z_0)}(h) = w$ together with its conjugate, so $L$ is bijective if and only if $\mathcal{J}$ is.

When $A$ is invertible, the second block row gives $\overline{h} = \overline{A}^{-1}(\overline{w} - \overline{B}h)$; substituting into the first equation gives $Sh = w - B\overline{A}^{-1}\overline{w}$, yielding (42) once $S$ is invertible. Conversely, any such $h$ solves $Ah + B\overline{h} = w$ by reversing the steps.

When $B$ is invertible, the first block row gives $\overline{h} = B^{-1}(w - Ah)$; substituting into the second equation gives $(\overline{B} - \overline{A}B^{-1}A)h = \overline{w} - \overline{A}B^{-1}w$, i.e., $Rh = \overline{w} - \overline{A}B^{-1}w$, yielding (43) once $R$ is invertible. The scalar identity $R = (|B|^2 - |A|^2)/B$ (for $n = 1$) confirms $R$ is invertible exactly when $|B| \neq |A|$. □

The two pivots are two ways of computing the same vector $\left[L_{(f,z_0)}\right]^{-1}(w)$. Whenever both apply they give the same NRW step, thus the same iterates – so the choice affects applicability, cost and rounding, but not the iteration.

While the A-pivot may seem natural because $B$ vanishes when $f$ is holomorphic around the root, the B-pivot has a simple interpretation. Regarding $f$ as a function of $\zeta = \overline{z}$ exchanges the roles of the Wirtinger operators, $\partial f/\partial\zeta = B$ and $\partial f/\partial\overline{\zeta} = A$, so pivoting on $B$ is essentially pivoting on $A$ with $\overline{z}$ as the unknown. In the extreme case that $f$ is purely a function of $\overline{z}$, i.e., $f(z,\overline{z}) = g(\overline{z})$ with $g$ holomorphic, $A = 0$ and the A-pivot is unavailable, while the B-pivot reduces to the classical Newton iteration $\overline{z}_{k+1} = \overline{z}_k - [g'(\overline{z}_k)]^{-1}g(\overline{z}_k)$ for $g$. We call this case as *anti-holomorphic* case.

When both pivots apply, the choice is not clear-cut and may come down to further analysis or experimentation; however, a "measure of holomorphy" can be discussed. Between the complete holomorphy and complete anti-holomorphy extremes, say that $L_{(f,z_0)}$ is *$A$-dominant* if $\sigma_{\min}(A) > \|B\|$ and *$B$-dominant* if $\sigma_{\min}(B) > \|A\|$, where $\sigma_{\min}$ denotes the smallest singular value. In the A-dominant case $A$ is invertible and

$$\left\|A^{-1}B\overline{A}^{-1}\overline{B}\right\| \leq \|B\|^2/\sigma_{\min}(A)^2 < 1, \tag{44}$$

so $S$ is invertible as well and the A-pivot applies. Symmetrically, the B-pivot applies in the B-dominant case. The A-pivot is thus natural for near-holomorphic maps and the B-pivot for near-antiholomorphic ones.

In terms of implementation, we show further in this paper that both pivots and the NRW iteration step can be implemented with a single formula. Therefore, in practical terms, choosing between the A and B pivots comes down to "measuring the holomorphy" around the sought root, or simply using the one that yields less wall-clock time for solving. They produce the exact same iterates, the same number of iterations, and all the convergence criteria are identica.

What lemma 3 establishes is: the widely-linear operator $L$ is invertible whenever $\mathcal{J}$ is; furthermore, if $A$ and $B$ are bounded and $L$ is invertible, then $L^{-1}$ is also bounded. Applying the triangle inequality to (42) and (43) respectively:

$$\left\|\left[L_{(f,z_0)}\right]^{-1}\right\| \leq \|S^{-1}\| + \left\|S^{-1}B\overline{A}^{-1}\right\| \quad \text{(A-pivot)}, \tag{45}$$

$$\left\|\left[L_{(f,z_0)}\right]^{-1}\right\| \leq \|R^{-1}\| + \|R^{-1}\overline{A}B^{-1}\| \quad \text{(B-pivot)}. \tag{46}$$

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*Remark* 3 (Scalar closed form of $\beta$). For $n = 1$, the realified Jacobian $J_R$ has singular values $|A| + |B|$ and $||A| - |B||$, so $\|J_R\| = |A| + |B|$ and $\left\|J_R^{-1}\right\| = 1/\,||A| - |B||$. Equivalently, with the A-pivot, $S = A - B\overline{A}^{-1}\overline{B} = (|A|^2 - |B|^2)/\overline{A}$, giving $\left\|S^{-1}\right\| = |A|/\left||A|^2 - |B|^2\right|$ and $\left\|S^{-1}B\overline{A}^{-1}\right\| = |B|/\left||A|^2 - |B|^2\right|$, so

$$\left\|\left[L_{(f,z_0)}\right]^{-1}\right\| = \frac{|A| + |B|}{||A|^2 - |B|^2|} = \frac{1}{||A| - |B||}, \tag{47}$$

and $\beta = 1/\,||A| - |B||$. The B-pivot gives the same value; in particular $\beta = 1/(|A| - |B|)$ when $|A| > |B|$ and $\beta = 1/(|B| - |A|)$ when $|B| > |A|$.

We now prove the equivalent Kantorovich theorem for the NRW method.

**Theorem 5** (Newton-Kantorovich theorem for the NRW method)**.** *Let $\Omega \subset \mathbb{C}^n$ be open and convex, let $f : \Omega \to \mathbb{C}^n$ be Lipschitz-Wirtinger-differentiable (Corollary 2), and let an initial guess $z_0 \in \Omega$ be such that $L_{(f,z_0)}$ is invertible in the sense of Lemma 3. Suppose the following conditions are satisfied:*

1. ***(K1): Bounded inverse.*** $\left\|\left[L_{(f,z_0)}\right]^{-1}\right\| \leq \beta$;
2. ***(K2): First step magnitude.*** $\left|\left[L_{(f,z_0)}\right]^{-1}(f(z_0))\right| \leq \eta$;
3. ***(K3): Lipschitz continuity of the approximation.*** $\left\|L_{(f,u)} - L_{(f,v)}\right\| \leq L\,|u - v|$ *for all* $u, v \in \Omega$.

*Define the Kantorovich constant $h = \beta L \eta$ and the majorant polynomial*

$$p(t) = \frac{\beta L}{2}t^2 - t + \eta. \tag{48}$$

*If $h \leq \dfrac{1}{2}$ and the closed ball $\overline{B}(z_0, t_*) \subset \Omega$, where*

$$t_* = \frac{1 - \sqrt{1 - 2h}}{\beta L}, \quad t_{**} = \frac{1 + \sqrt{1 - 2h}}{\beta L} \tag{49}$$

*are the roots of $p$, then:*

*(i) the NRW iterates*

$$z_{k+1} = z_k - \left[L_{(f,z_k)}\right]^{-1}(f(z_k)) \tag{50}$$

*are well defined (each $L_{(f,z_k)}$ is invertible), remain in $\overline{B}(z_0, t_*)$, and satisfy*

$$|z_{k+1} - z_k| \leq t_{k+1} - t_k, \tag{51}$$

*where $t_0 = 0$ and $t_{k+1} = t_k - p(t_k)/p'(t_k)$ is the Newton sequence of the majorant, which increases to $t_*$;*

*(ii) the sequence $(z_k)$ converges to a point $z_* \in \overline{B}(z_0, t_*)$ with $f(z_*) = 0$, and*

$$|z_* - z_k| \leq t_* - t_k \quad \text{for all } k; \tag{52}$$

*(iii) if moreover $\overline{B}(z_0, r) \subset \Omega$ for some $t_* \leq r \leq t_{**}$, then $z_*$ is the only root of $f$ in $\overline{B}(z_0, t_*) \cup B(z_0, r)$;*

*(iv) if $h < \dfrac{1}{2}$, convergence is quadratic:*

$$|z_* - z_{k+1}| \leq \frac{\beta L}{2\sqrt{1 - 2h}}\,|z_* - z_k|^2, \tag{53}$$

*and, with $\theta = t_*/t_{**} < 1$,*

$$|z_* - z_k| \leq t_* - t_k = (t_{**} - t_*)\,\frac{\theta^{2^k}}{1 - \theta^{2^k}}; \tag{54}$$

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*if* $h = \frac{1}{2}$*, convergence is linear with ratio* $\frac{1}{2}$*:*

$$|z_* - z_k| \leq t_* - t_k = \eta\, 2^{1-k}. \tag{55}$$

*Remark* 4. The norm $\|\cdot\|$ in (K1) and (K3) is the induced operator norm of the widely-linear map $h \mapsto Ah + B\overline{h}$, computed with respect to the Euclidean norm on $\mathbb{C}^n$. For $n = 1$ this reduces to $\|L\| = |A| + |B|$.

*Remark* 5. The majorant $p(t) = \frac{\beta L}{2}t^2 - t + \eta$ is a rescaled version of the one in Theorem 1: multiplying the traditional polynomial $\frac{L}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta}$ by $\beta$ yields $p(t)$ above. Both share the same roots and Newton sequences; the rescaling simplifies notation throughout the proof.

*Proof.* For simplicity of notation, write $L_k = L_{(f,z_k)}$, $\Delta_k = z_{k+1} - z_k$, and $f_k = f(z_k)$.

*The majorant sequence.* On $[0, t_*)$ one has $p > 0$ and

$$p'(t) = \beta L t - 1 \leq \beta L t_* - 1 = -\sqrt{1 - 2h} \leq 0, \tag{56}$$

with strict negativity for $t < t_* \leq 1/(\beta L)$. If $t_k \in [0, t_*)$, then

$$t_{k+1} = t_k + \frac{p(t_k)}{(1 - \beta L t_k)} \geq t_k, \tag{57}$$

and since $p$ is convex, its tangent at $t_k$ lies below $p$, so the tangent's zero $t_{k+1}$ does not exceed the smaller root: $t_{k+1} \leq t_*$. Hence $(t_k)$ is nondecreasing and bounded by $t_*$; its limit is a fixed point of the Newton map, i.e., a root of $p$ in $[0, t_*]$, hence equals $t_*$. Two identities will be used repeatedly: since $t_{k+1}$ annihilates the linearization of $p$ at $t_k$,

$$p(t_{k+1}) = p(t_{k+1}) - p(t_k) - p'(t_k)(t_{k+1} - t_k) = \frac{\beta L}{2}(t_{k+1} - t_k)^2, \tag{58}$$

and consequently

$$t_{k+2} - t_{k+1} = \frac{p(t_{k+1})}{1 - \beta L t_{k+1}} = \frac{\beta L (t_{k+1} - t_k)^2}{2(1 - \beta L t_{k+1})}. \tag{59}$$

*Induction.* Assume the three inductive hypotheses:

- $(a_k)$: $z_k$ is defined with $|z_k - z_0| \leq t_k$;
- $(b_k)$: $L_k$ is invertible with $\left\|[L_k]^{-1}\right\| \leq \dfrac{\beta}{1 - \beta L t_k}$;
- $(c_k)$: $|\Delta_k| \leq t_{k+1} - t_k$.

For the initial step $k = 0$, we see $(a_0)$ is trivial, $(b_0)$ is (K1), and $(c_0)$ is (K2) with $t_1 - t_0 = \eta$. We thus assume $(a_i)$–$(c_i)$ for $i \leq k$. Then

$$|z_{k+1} - z_0| \leq \sum_{i \leq k} |\Delta_i| \leq t_{k+1} \leq t_*, \tag{60}$$

giving $(a_{k+1})$. For $(b_{k+1})$, (K1) and (K3) give

$$\left\|[L_0]^{-1}(L_0 - L_{k+1})\right\| \leq \beta L t_{k+1} < \beta L t_* \leq 1, \tag{61}$$

so the Neumann series for $[L_{k+1}]^{-1} = [L_0 - (L_0 - L_{k+1})]^{-1}$ gives

$$\left\|[L_{k+1}]^{-1}\right\| \leq \frac{\beta}{1 - \beta L t_{k+1}}. \tag{62}$$

For $(c_{k+1})$, we bound the new residual. Because $\Delta_k$ solves $f_k + L_k(\Delta_k) = 0$, Corollary 3 gives

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$$f_{k+1} = f_{k+1} - f_k - L_k(\Delta_k) = \int_0^1 \left(L_{(f,z_k+t\Delta_k)} - L_k\right)(\Delta_k)\,dt, \tag{63}$$

where the segment $z_k + t\Delta_k$ remains in $\overline{B}(z_0, t_{k+1}) \subset \overline{B}(z_0, t_*) \subset \Omega$. By (K3), $\left\|L_{(f,z_k+t\Delta_k)} - L_k\right\| \leq Lt\,|\Delta_k|$, so

$$|f_{k+1}| \leq \int_0^1 Lt\,|\Delta_k|^2\,dt = \frac{L}{2}\,|\Delta_k|^2 \leq \frac{L}{2}\,(t_{k+1} - t_k)^2. \tag{64}$$

Therefore, by $(b_{k+1})$ and (59),

$$|\Delta_{k+1}| = \left|[L_{k+1}]^{-1}(f_{k+1})\right| \leq \frac{\beta}{1 - \beta L t_{k+1}} \cdot \frac{L}{2}\,(t_{k+1} - t_k)^2 = t_{k+2} - t_{k+1}, \tag{65}$$

closing the induction.

*Convergence and existence of a root.* For $m > k$, $|z_m - z_k| \leq t_m - t_k$; since $(t_k)$ converges, $(z_k)$ is Cauchy and converges to some $z_*$ with $|z_* - z_k| \leq t_* - t_k$, which is (52) and places $z_*$ in $\overline{B}(z_0, t_*)$. From (64), $f_k \to 0$ since $t_{k+1} - t_k \to 0$, and continuity of $f$ gives $f(z_*) = 0$.

*Uniqueness.* Let $y_*$ be any root of $f$ in the stated region; if $y_* = z_0$, then $\eta$ may be taken 0 and everything collapses to $z_* = z_0$, so assume $y_* \neq z_0$, and note that in every case $q = |y_* - z_0|\,/t_{**} < 1$ (for $y_* \in \overline{B}(z_0, t_*)$ with $t_* < t_{**}$ this is immediate; for $y_* \in B(z_0, r)$, $r \leq t_{**}$, it is the strict inequality of the open ball; and if $t_* = t_{**}$, i.e., $h = \frac{1}{2}$, the region is $\overline{B}(z_0, t_*)$ and one replaces $q < 1$ below by $q \leq 1$ together with $t_{**} - t_k \to 0$). Since $f(y_*) = 0$ and $f_k + L_k(\Delta_k) = 0$,

$$\begin{aligned} y_* - z_{k+1} &= y_* - z_k - [L_k]^{-1}(f(y_*) - f_k) \\ &= -[L_k]^{-1} \int_0^1 \left(L_{(f,z_k+t(y_*-z_k))} - L_k\right)(y_* - z_k)\,dt, \end{aligned} \tag{66}$$

whence, exactly as in (64), using $(b_k)$ and (K3),

$$|y_* - z_{k+1}| \leq \frac{\beta L\,|y_* - z_k|^2}{2(1 - \beta L t_k)}. \tag{67}$$

Expanding $p(t_{**}) = 0$ around $t_k$ gives $p(t_k) + p'(t_k)(t_{**} - t_k) = -\dfrac{\beta L}{2}(t_{**} - t_k)^2$, hence the exact identity

$$t_{**} - t_{k+1} = \frac{\beta L\,(t_{**} - t_k)^2}{2(1 - \beta L t_k)}. \tag{68}$$

We claim $|y_* - z_k| \leq q^{2^k}(t_{**} - t_k)$: for $k = 0$ this is the definition of $q$; assuming it for $k$, (67) and (68) give

$$|y_* - z_{k+1}| \leq q^{2^{k+1}}\,\frac{\beta L\,(t_{**} - t_k)^2}{2(1 - \beta L t_k)} = q^{2^{k+1}}\,(t_{**} - t_{k+1}). \tag{69}$$

If $q < 1$, the right side tends to 0 since $t_{**} - t_k \leq t_{**}$; if $q = 1$ (possible only when $h = \frac{1}{2}$), it tends to 0 because $t_{**} - t_k = t_* - t_k \to 0$. In either case $y_* = \lim z_k = z_*$.

*Rates.* Taking $y_* = z_*$ in (67) and bounding

$$1 - \beta L t_k \geq 1 - \beta L t_* = \sqrt{1 - 2h} \tag{70}$$

yields (53) when $h < \frac{1}{2}$. For the closed form (54), factor $p(t) = \dfrac{\beta L}{2}(t - t_*)(t - t_{**})$ and compute, for the Newton map $N(t) = t - p(t)\,/p'(t)$,

$$N(t) - t_* = \frac{\frac{\beta L}{2}(t - t_*)^2}{p'(t)}, \quad N(t) - t_{**} = \frac{\frac{\beta L}{2}(t - t_{**})^2}{p'(t)}, \tag{71}$$

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so that the ratio $u_k = (t_* - t_k)\,/\,(t_{**} - t_k)$ obeys the squaring law $u_{k+1} = u_k^2$, with $u_0 = t_*/t_{**} = \theta$; solving $u_k = \theta^{2^k}$ for $t_* - t_k$ gives (54). When $h = \frac{1}{2}$, one has $t_* = t_{**} = 2\eta$, $p\,(t) = \dfrac{\beta L}{2}\,(t - t_*)^2$, and the Newton map halves the distance to the double root: $t_* - t_{k+1} = \dfrac{1}{2}\,(t_* - t_k)$, whence $t_* - t_k = t_*\,2^{-k} = \eta\,2^{1-k}$. □

4.2. **Affine-covariant semilocal convergence.** Theorem 5 bounds the inverse $\left[L_{(f,z_0)}\right]^{-1}$ and the variation of $z \mapsto L_{(f,z)}$ separately, through $\beta$ and $L$. Only their composition enters the Newton iteration, and the norm of the composition can be much smaller than the product of the norms. This happens when the directions in which $\left[L_{(f,z_0)}\right]^{-1}$ is large are not the directions in which $L_{(f,z)}$ varies. Following Deuflhard [11], the remedy is to bound the composition directly. In the present setting the composition is again a widely-linear operator: $\left[L_{(f,z_0)}\right]^{-1}$ is real-linear, so $\left[L_{(f,z_0)}\right]^{-1} \circ L_{(f,u)}$ is real-linear, and hence widely-linear by Lemma 2. The resulting theorem is therefore stated and checked natively in $\mathbb{C}^n$, just like Theorem 5.

**Theorem 6** (Affine-covariant Newton-Kantorovich theorem for the NRW method)**.** *Let $\Omega \subset \mathbb{C}^n$ be open and convex, let $f : \Omega \to \mathbb{C}^n$ be Wirtinger differentiable on $\Omega$, and let $z_0 \in \Omega$ be such that $L_0 \coloneqq L_{(f,z_0)}$ is invertible in the sense of Lemma 3. Suppose that*

(1) ***(D1): First step magnitude.*** *$\left|L_0^{-1}\,(f\,(z_0))\right| \le \eta$;*
(2) ***(D2): Affine-covariant Lipschitz condition.*** *$\left\|L_0^{-1} \circ \left(L_{(f,u)} - L_{(f,v)}\right)\right\| \le \omega\,|u - v|$ for all $u, v \in \Omega$.*

*Let $h = \omega\eta$ and $t_*, t_{**} = \left(1 \mp \sqrt{1 - 2h}\right)/\omega$. If $h \le \frac{1}{2}$ and $\overline{B}\,(z_0, t_*) \subset \Omega$, then conclusions (i)–(iv) of Theorem 5 hold with $\beta L$ replaced by $\omega$ throughout, that is, with the majorant $p(t) = \frac{\omega}{2}t^2 - t + \eta$.*

*Proof.* Consider $F = L_0^{-1} \circ f : \Omega \to \mathbb{C}^n$. Since $L_0^{-1}$ is a bounded real-linear map,

$$F\,(z + k) - F\,(z) - L_0^{-1} \circ L_{(f,z)}\,(k) = L_0^{-1}\left(f\,(z + k) - f\,(z) - L_{(f,z)}\,(k)\right) = o\,(|k|)\,. \tag{72}$$

The map $L_0^{-1} \circ L_{(f,z)}$ is widely-linear, so by the uniqueness part of Theorem 3, $F$ is Wirtinger differentiable with $L_{(F,z)} = L_0^{-1} \circ L_{(f,z)}$. By (D2), $z \mapsto L_{(F,z)}$ is Lipschitz with constant $\omega$, so $F$ is Lipschitz-Wirtinger-differentiable (Corollary 2). Now check the hypotheses of Theorem 5 for $F$. Since $L_{(F,z_0)}$ is the identity, (K1) holds with $\beta = 1$. (K2) is (D1), and (K3) is (D2) with $L = \omega$. Hence the Kantorovich constant of $F$ is $1 \cdot \omega \cdot \eta = h$, and Theorem 5 applies to $F$. Finally, the NRW iterates of $F$ and $f$ coincide, because

$$\left[L_{(F,z)}\right]^{-1}(F\,(z)) = \left[L_0^{-1} \circ L_{(f,z)}\right]^{-1}\left(L_0^{-1}\,(f\,(z))\right) = \left[L_{(f,z)}\right]^{-1}(f\,(z))\,,$$

and the roots of $F$ and $f$ coincide because $L_0^{-1}$ is injective. □

Theorem 6 is never weaker than Theorem 5. Since $\left\|L_0^{-1} \circ (L_{(f,u)} - L_{(f,v)})\right\| \le \beta\left\|L_{(f,u)} - L_{(f,v)}\right\|$, every pair $(\beta, L)$ satisfying (K1) and (K3) gives $\omega \le \beta L$. Theorem 5 is therefore a special case of Theorem 6 obtained with the bound $\omega = \beta L$, and every initial guess it certifies is also certified here. Because $t_* = 2\eta/(1 + \sqrt{1 - 2h})$ increases with $h$, a smaller $\omega$ also gives a smaller existence radius $t_*$ and a larger uniqueness radius $t_{**}$.

The new condition is also invariant under a larger class of transformations. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be any invertible widely-linear map, $T(w) = Pw + Q\overline{w}$, independent of $z$. Then $L_{(T \circ f,z)} = T \circ L_{(f,z)}$, so $T \circ f$ has the same NRW iterates as $f$ and the same $\eta$, $\omega$ and $h$. The certificate of Theorem 6 is therefore invariant under complex-linear recombinations of the equations, and also under conjugating some or all of them or mixing $f$ with $\overline{f}$. This group is larger than the complex-linear one of the holomorphic theory. The constants $\beta$ and $L$ of Theorem 5, in contrast, change under such transformations, although their product never falls below $\omega$.

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The gain is not automatic. In the scalar case with a constant $\partial f/\partial z = A$, as in the SGIB example of Section 6, one has $L_{(f,u)} - L_{(f,v)} = (B(u) - B(v))\,\overline{(\cdot)}$. Conjugation is an isometry, so $\left\| L_0^{-1} \circ \left(c\,\overline{(\cdot)}\right) \right\| = |c|\,\left\|L_0^{-1}\right\|$ for every $c \in \mathbb{C}$. Hence $\omega = \beta L$ exactly, when both are taken as the smallest constants on the same set, and the two theorems give the same certificate. A gain requires several dimensions, in which the worst direction of $L_0^{-1}$ and the directions in which $L_{(f,z)}$ varies can differ. Section 7 shows that this gain can amount to orders of magnitude.

The constant $\omega$ in (D2) is a supremum over all pairs $u, v \in \Omega$ and is rarely available in closed form. The iteration itself, however, reveals part of it. Let $\Delta_0 = -L_0^{-1}(f(z_0))$ be the first NRW step, so that $z_1 = z_0 + \Delta_0$. If $f$ coincided with its widely-linear approximation at $z_0$, the residual $f(z_1)$ would vanish; it is nonzero only because $L_{(f,z)}$ changes along the segment from $z_0$ to $z_1$. To measure this change in the affine-covariant norm of (D2), consider the simplified Newton correction $\tilde{\Delta}_1 = -L_0^{-1}(f(z_1))$. It differs from the next NRW step $\Delta_1 = -L_1^{-1}(f(z_1))$ only in that it applies the inverse at $z_0$ instead of $z_1$; the tilde marks this. It reuses the factorization of $L_0$ and therefore costs only one additional solve. Since $f(z_0) + L_0(\Delta_0) = 0$, Corollary 3 gives $\tilde{\Delta}_1 = -\int_0^1 L_0^{-1} \circ \left(L_{(f,z_0+t\Delta_0)} - L_0\right)(\Delta_0)\,dt$, and (D2) bounds the integrand by $\omega t\,|\Delta_0|^2$. Hence $\left|\tilde{\Delta}_1\right| \le \frac{\omega}{2}\,|\Delta_0|^2$, that is,

$$\hat{\omega}_0 := \frac{2\left|\tilde{\Delta}_1\right|}{|\Delta_0|^2} \le \omega. \tag{73}$$

In words, $\hat{\omega}_0$ is the smallest value of $\omega$ consistent with what the first step has observed. It is only a lower bound, because (D2) must hold for every pair $u, v \in \Omega$ and every direction, whereas $\hat{\omega}_0$ samples a single segment and a single direction. The same construction at step $k$, with $\Delta_k = z_{k+1} - z_k$ and $\tilde{\Delta}_{k+1} = -L_k^{-1}(f(z_{k+1}))$, gives $\hat{\omega}_k = 2\left|\tilde{\Delta}_{k+1}\right| / |\Delta_k|^2$, a lower bound for the affine-covariant constant relative to $L_k = L_{(f,z_k)}$. Together with the contraction factors $\Theta_k = \left|\tilde{\Delta}_{k+1}\right| / |\Delta_k|$, these are the classical computational estimates of [11]. They cannot certify convergence on their own, but they indicate how much sharper Theorem 6 can be than Theorem 5 for a given problem.

4.3. **Local convergence analysis.** As in the traditional case, when an initial guess is already known to be close to a root, the NRW method preserves its characteristically rapid rate of convergence, without requiring the semilocal hypotheses (K1)–(K3) of Theorem 5.

**Theorem 7** (Quadratic convergence of the NRW method)**.** *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be Wirtinger differentiable on $\Omega$. Assume there exists a point $z^* \in \Omega$ such that $f(z^*) = 0$ and $L_{(f,z^*)}$ is invertible in the sense of Lemma 3. Furthermore, assume $z \mapsto L_{(f,z)}$ is locally Lipschitz continuous in a neighborhood of $z^*$. Then, for $z_0$ sufficiently close to $z^*$, the error sequence $e_n = z_n - z^*$ generated by the NRW iteration satisfies*

$$|e_{n+1}| \le C\,|e_n|^2 \tag{74}$$

*for a real positive $C$ independent of $n$.*

*Proof.* By the definition of the NRW iteration, we subtract $z^*$ from both sides:

$$z_{n+1} - z^* = z_n - z^* - \left[L_{(f,z_n)}\right]^{-1}(f(z_n)) \tag{75}$$

Factoring out the inverse derivative operator yields

$$e_{n+1} = \left[L_{(f,z_n)}\right]^{-1}\left(L_{(f,z_n)}(e_n) - f(z_n)\right) \tag{76}$$

Noting that $f(z^*) = 0$, Corollary 3 expresses $f(z_n)$ as an integral operator along the segment from $z^*$ to $z_n$:

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$$f(z_n) = f(z_n) - f(z^*) = \int_0^1 L_{(f,z^*+t(z_n-z^*))}(e_n)\,\mathrm{d}t \tag{77}$$

Substituting this back into the error expression gives

$$e_{n+1} = \left[L_{(f,z_n)}\right]^{-1} \int_0^1 \left(L_{(f,z_n)} - L_{(f,z^*+t(z_n-z^*))}\right)(e_n)\,\mathrm{d}t \tag{78}$$

Taking the norm of both sides and applying the local Lipschitz constant $L$ of $z \mapsto L_{(f,z)}$ yields

$$|e_{n+1}| \le \left\|\left[L_{(f,z_n)}\right]^{-1}\right\| \int_0^1 L\,|z_n - (z^* + t\,(z_n - z^*))|\,|e_n|\,\mathrm{d}t \tag{79}$$

$$= \left\|\left[L_{(f,z_n)}\right]^{-1}\right\| \int_0^1 L\,(1-t)\,|e_n|^2\,\mathrm{d}t \tag{80}$$

$$= \frac{L\left\|\left[L_{(f,z_n)}\right]^{-1}\right\|}{2}\,|e_n|^2 \tag{81}$$

Let $\beta^* = \left\|\left[L_{(f,z^*)}\right]^{-1}\right\|$. We restrict to the neighborhood $|e_n| \le \frac{1}{2\beta^* L}$, which is non-empty since $z_0$ is assumed sufficiently close to $z^*$. In this region the Banach perturbation lemma applies: since $\left\|L_{(f,z_n)} - L_{(f,z^*)}\right\| \le L\,|e_n| \le \frac{1}{2\beta^*}$, we have

$$\left\|\left[L_{(f,z_n)}\right]^{-1}\right\| \le \frac{\beta^*}{1 - \beta^* L\,|e_n|} \le 2\beta^*. \tag{82}$$

By induction, each iterate $z_n$ remains in this neighborhood (since $C\,|e_0| < 1$ with $C = \beta^* L$), so the bound holds for all $n$. Therefore

$$|e_{n+1}| \le \frac{L \cdot 2\beta^*}{2}\,|e_n|^2 = \beta^* L\,|e_n|^2 = C\,|e_n|^2. \tag{83}$$

□

Thus local quadratic convergence of the NRW method is established, generalizing the traditional NR result to the larger class of non-holomorphic problems: once an iterate lands close enough to a simple root — one where $L_{(f,z^*)}$ is invertible — convergence is quadratic with no dependence on the global bounds $\beta, \eta$ of Theorem 5.

## 5. Numerical implementation

The numerical implementation of the NRW is, as is the traditional NR, a sequential implementation of its steps (38). The widely-linear equation $Ah + B\overline{h} = -f$ already constitutes $2n$ real equations in $2n$ real unknowns ($h$ and $\overline{h}$ are not independent), so the system is not underdetermined. Nonetheless, it is convenient to work in the augmented form below, which recasts the system as a standard linear solve over $\mathbb{C}^{2n}$.

We use the augmented function $g(z) = \left[f(z), \overline{f}(z)\right]^\intercal$, which is defined for any $f : \mathbb{C}^n \to \mathbb{C}^m$ (its Wirtinger Jacobian below being $2m \times 2n$), although the Schur reduction that follows requires the square case $m = n$; and the first-order approximation of this augmentation yields

$$g(z_k + \Delta z) = g(z_k) + \left[\frac{\partial g}{\partial z}\right]_{z_k} \begin{bmatrix} \Delta z \\ \Delta \overline{z} \end{bmatrix} + O\left(|\Delta z|^2\right) \tag{84}$$

where

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$$\frac{\partial g}{\partial z} = \begin{bmatrix} \dfrac{\partial f}{\partial z} & \dfrac{\partial f}{\partial \overline{z}} \\ \dfrac{\partial \overline{f}}{\partial z} & \dfrac{\partial \overline{f}}{\partial \overline{z}} \end{bmatrix} = \begin{bmatrix} H_f & W_f \\ \overline{W_f} & \overline{H_f} \end{bmatrix} = \mathcal{J}, \tag{85}$$

since $\partial\overline{f}/\partial z = \overline{\partial f/\partial \overline{z}}$ and $\partial\overline{f}/\partial\overline{z} = \overline{\partial f/\partial z}$. That is, the Wirtinger Jacobian of the augmented function is precisely the augmented matrix $\mathcal{J}$ of Lemma 3, with $A = H_f$ and $B = W_f$ (see Section 3.2 for this notation). This yields an approximation for a root given by $(z_k + \Delta z, \overline{z_k} + \Delta\overline{z})$ where

$$0 = g(z_k) + \left[\frac{\partial g}{\partial z}\right]_{z_k} \begin{bmatrix} \Delta z \\ \Delta \overline{z} \end{bmatrix} \tag{86}$$

therefore, this defines a first crude algorithm for the NRW step as

$$\begin{bmatrix} \Delta z \\ \Delta \overline{z} \end{bmatrix} = -\left[\mathcal{J}(z_k)\right]^{-1} g(z_k) \tag{87}$$

which requires a $2n$-dimension inversion.

5.1. **Schur reduction.** We can reduce the double dimensionality of (86) in $\mathbb{C}^{(2n)}$ back to $\mathbb{C}^n$ via Schur reduction and Lemma 3. Recall the notation $H_f = \partial f/\partial z$ and $W_f = \partial f/\partial\overline{z}$ of Section 3.2. By applying Schur reduction to $\partial g/\partial z$ one obtains

$$\begin{bmatrix} z \\ \overline{z} \end{bmatrix}_{(k+1)} = \begin{bmatrix} z \\ \overline{z} \end{bmatrix}_k - \begin{bmatrix} \overline{(\Sigma^{-1})} & \overline{(\Sigma^{-1})}\,\overline{\Gamma} \\ \Sigma^{-1}\,\Gamma & \Sigma^{-1} \end{bmatrix} \begin{bmatrix} f(z_k) \\ \overline{f}(z_k) \end{bmatrix} \tag{88}$$

where

$$\begin{aligned} &\text{A-pivot:} \quad \Gamma = -\overline{W_f}\,H_f^{-1}, \quad \Sigma = \overline{H_f} + \Gamma W_f, \\ &\text{B-pivot:} \quad \Gamma = -\overline{H_f}\,W_f^{-1}, \quad \Sigma = \overline{W_f} + \Gamma H_f. \end{aligned} \tag{89}$$

With the A-pivot, $\Sigma = \overline{H_f} - \overline{W_f}H_f^{-1}W_f$ is the complex conjugate of the Schur complement $S = A - B\overline{A}^{-1}\overline{B}$ of Lemma 3, that is, $\Sigma = \overline{S}$. The B-pivot is the same construction with the roles of $H_f$ and $W_f$ exchanged, which by the discussion following Lemma 3 amounts to taking $\overline{z}$ as the unknown; then $\overline{\Sigma} = W_f - H_f\overline{W_f}^{-1}\overline{H_f} = \overline{R}$, with $R$ the reduced matrix of the same lemma. The process therefore depends on the invertibility of the pivot block, $H_f$ or $W_f$, and of $\Sigma$. If these matrices are invertible, separating both lines of (88) yields

$$z_{(k+1)} = z_k - \overline{\Sigma^{-1}}\left[f(z_k) + \overline{\Gamma}\,\overline{f}(z_k)\right] \tag{90}$$

$$\overline{z}_{(k+1)} = \overline{z}_k - \Sigma^{-1}\left[\overline{f}(z_k) + \Gamma\, f(z_k)\right] \tag{91}$$

which are essentially the same equation. With the A-pivot, the first line is the update of $z$. With the B-pivot, the two lines exchange roles: the first line gives $\overline{z}_{(k+1)}$, so the iteration is naturally carried out in $\overline{z}$, and $z$ is recovered by conjugating the final iterate.

Therefore, in practical terms, the A-pivot and B-pivot iterations can be built using the same procedure and code, with two small caveats. When using the B-pivot, the matrices $H_f$ and $W_f$ are swapped, and since in this pivot the root found is $\overline{z}$, at the end of iterations after tolerance is reached one needs to conjugate the result.

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5.2. **CBLAS and LAPACKE implementation.** The iteration (91) requires calculating several matrix inverses such as $\Sigma^{-1}$ and $H_f^{-1}$ which needs to be done carefully. We use the BLAS/LAPACKE implementations in C language to execute the algorithm; the implementation is carried out in double-precision complex types `lapack_complex_double`. The LAPACKE functions used are:

(1) `zgesv(A,B)`: takes two pointers to a complex type for matrices $A$ and $B$ and stores the solution of $AX = B$ into a likewise pointer for $x$. The solution is computed via the LU decomposition of $A$. If the matrix problem is "flipped", that is, we want to calculate $XA = B$, the equivalent transposed system $A^\intercal X^\intercal = B^\intercal$ is calculated by using flipped layout flags: since a row-major matrix is the exact transpose of its column-major counterpart, one calls `zgesv()` with flipped layout flags, that is, $A, B, X$ with `LAPACK_COL_MAJOR`;
(2) `zgemm(`$\alpha$`,A,B,`$\beta$`)`: general dense-matrix multiplication. Computes $C \leftarrow \alpha AB + \beta C$;
(3) `zgemv(`$\alpha$`,A,x,`$\beta$`)`: general matrix-by-vector multiplication. Computes $y \leftarrow \alpha Ax + \beta y$;
(4) `zaxpy(`$\alpha$`,x)`: scalar-by-vector multiplication. Computes $y \leftarrow \alpha x + y$.

We thus rewrite the NRW update equation as

$$\overline{\Sigma}\,\Delta z_k = -\left[f(z_k) + \overline{\Gamma}\,\overline{f}(z_k)\right] = b_k \tag{92}$$

and we have now transformed the problem to an equivalent $Ax = b$. The objective now is to also break down the calculations of $\Gamma$ and $\Sigma$ into problems of the type $AX = B$. Because $\Sigma$ is given in terms of $\Gamma$, we first analyze how to calculate $\Gamma$. By (89),

$$\Gamma H_f = -\overline{W_f} \tag{93}$$

and $\Gamma$ is calculated using `zgesv` with the flipped layout flags trick. Then, for $\Sigma$, by definition (89),

$$\Sigma = \overline{H_f} + \Gamma W_f \tag{94}$$

which is simple to get using the general dense matrix multiplication `zgemm`. Then we use the general LU factorization `zgesv` to solve for $\Delta z_k$ through (92). The algorithm 1 shows a pseudocode of this procedure. Both pivots of (89) use the same algorithm: for the B-pivot, $H_f$ and $W_f$ are passed in exchanged order, and the result is then the step in $\overline{z}$. Rather than conjugating this step at every iteration, the B-pivot iteration stores $w_k = \overline{z_k}$ and updates $w_{k+1} = w_k + \Delta z_k$ with the output of the algorithm; only the final iterate is conjugated to recover $z$. Since $f$, $H_f$ and $W_f$ depend on both $z$ and $\overline{z}$, their evaluation is unaffected by which of the two is stored. If the pivot is changed during the iteration, the stored iterate is conjugated once at the switch.

**Algorithm 1** NRW step pseudocode using CBLAS/LAPACKE.

**Require:** $f, H_f, W_f$ ▷ For the B-pivot, pass $W_f$ in place of $H_f$ and $H_f$ in place of $W_f$
$\overline{H_f} \leftarrow$ `conj` $(H_f)$
$B_f \leftarrow -$`conj` $(W_f)$ ▷ Conjugation and negation in a single pass over $W_f$
$\Gamma \leftarrow$ `zgesv` $(H_f, B_f)$ ▷ Solve (93). `zgesv` is called with flipped layout flags.
$\Sigma \leftarrow \overline{H_f}$ ▷ Initialize $\Sigma$ as $\overline{H_f}$, saving a declaration
$\Sigma \leftarrow$ `zgemm` $(1, \Gamma, W_f, 1)$ ▷ Computes $\Sigma \leftarrow 1 \times \Gamma \times W_f + 1 \times \Sigma$.
$\overline{\Sigma} \leftarrow$ `conj` $(\Sigma)$
$b \leftarrow$ `zgemv` $(-1, \Gamma, f, 0)$ ▷ $b \leftarrow -\Gamma f$; negation absorbed into `zgemv` scalar
$b \leftarrow$ `conj` $(b)$ ▷ $b \leftarrow -\overline{\Gamma f}$
$b \leftarrow$ `zaxpy` $(-1, f)$ ▷ $b \leftarrow -\overline{\Gamma f} - f = -(f + \overline{\Gamma f})$
$\Delta z_k \leftarrow$ `zgesv` $\left(\overline{\Sigma}, b\right)$ ▷ With the B-pivot, this is the step in $\overline{z}$

The conjugations in Algorithm 1 follow from (89) and the Schur reduction above. As an independent check, Remark 2 provides a verification oracle: the NRW iterates must coincide, up to

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rounding, with those of the classical Newton-Raphson method applied to the realified system. This was verified iteration by iteration for the example of Section 6.3 (Table 4).

This brings the total per-iteration cost of the method to

$$\underbrace{\tfrac{4}{3}n^3}_{\text{L3}} + \underbrace{n^3}_{\text{L5}} + \underbrace{n^2}_{\text{L7}} + \underbrace{n}_{\text{L9}} + \underbrace{\tfrac{1}{3}n^3 + n^2}_{\text{L10}} = \frac{8}{3}n^3 + 2n^2 + n \tag{95}$$

complex multiply-adds.

5.3. **The pure Newton step.** When one of the Wirtinger operators vanishes identically, that is, when $f$ is holomorphic ($W_f = 0$) or antiholomorphic ($H_f = 0$), the full NRW machinery – computing $\Gamma$, $\Sigma$, $\overline{\Sigma}$, and the combined right-hand side $b$ – is unnecessary. Theoretically, one could go on by setting the vanishing operator to $\mathbf{0}_{(n\times n)}$ in Algorithm 1; this would however make the machine perform unnecessary `FLOP`s and introduce floating-point errors. In the holomorphic case the step reduces to solving the single linear system

$$H_f\,(z_k)\,\Delta z_k = f\,(z_k) \tag{96}$$

and updating $z_{k+1} = z_k - \Delta z_k$, which is the classical Newton-Raphson method in $z$. In the antiholomorphic case $f(z) = g(\overline{z})$ with $g$ holomorphic and $W_f(z) = g'(\overline{z})$, so both $f$ and $W_f$ depend on $z$ only through $w = \overline{z}$. As noted after Lemma 3, the iteration is then the classical Newton-Raphson method for $g$ in $w$,

$$W_f\,\Delta w_k = f, \qquad w_{k+1} = w_k - \Delta w_k, \qquad w_0 = \overline{z_0},$$

which is (96) with $W_f$ in place of $H_f$ and $w$ in place of $z$. No conjugation is needed during the iterations: only the final iterate is conjugated to recover $z$. The sign convention deserves attention: rather than negating the right-hand side to form $-f$ and then adding $\Delta z_k$ to $z_k$, we solve directly for $\Delta z_k$ with $f$ as the right-hand side and subtract, absorbing the negation into the update sign at zero cost. Algorithm 2 shows the pseudocode.

**Algorithm 2** Pure Newton step pseudocode as implemented in the library.

**Require:** $f, H_f$ ▷ Antiholomorphic case: pass $W_f$ in place of $H_f$, and $z$ holds $w = \overline{z}$
1: $\Delta z \leftarrow f$ ▷ Copy $f$ into the solve buffer; no negation
2: $\Delta z \leftarrow \texttt{zgesv}\,(H_f, \Delta z)$ ▷ Solve $H_f \Delta z = f$
3: $z \leftarrow \texttt{zaxpy}\,(-1, \Delta z)$ ▷ $z \leftarrow z - \Delta z$; negation absorbed into scalar

This brings the total per-iteration cost of the pure Newton step to

$$\underbrace{\tfrac{1}{3}n^3 + n^2}_{\text{Line 2}} + \underbrace{n}_{\text{Line 3}} = \frac{1}{3}n^3 + n^2 + n \tag{97}$$

complex multiply-adds, thus substantially cheaper than the full NRW step (95).

The same algorithm also computes the "general NRW" step (87), without pivoting on either Wirtinger operator. By (86), the NRW step solves a linear system whose matrix is the augmented Wirtinger Jacobian $\mathcal{J}$ and whose right-hand side is the augmented function $g = \left[f, \overline{f}\right]^{\mathsf{T}}$. Algorithm 2 with $\mathcal{J}$ in place of $H_f$, $g$ in place of $f$ and $[z, \overline{z}]^{\mathsf{T}}$ in place of $z$ therefore performs the NRW step: the first $n$ components of the solution are the step in $z$, and the last $n$ are their conjugates. For dense operators this costs $\frac{1}{3}(2n)^3 = \frac{8}{3}n^3$ complex multiply-adds, the same leading cost as Algorithm 1, but it requires only that $\mathcal{J}$ be invertible, which by Lemma 3 is exactly the invertibility of the widely-linear approximation. When $H_f$ and $W_f$ are sparse, a sparse factorization of $\mathcal{J}$ preserves their sparsity; in the Gross-Pitaevskii application of Section 7 a suitable ordering of the unknowns makes $\mathcal{J}$ banded, and a banded LU factorization is used. This route is preferable when neither $H_f$ nor $W_f$ is cheap to invert or uniformly well conditioned.

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5.4. **A note on complex versus real `FLOPs`.** Conventionally, the term Floating-Point Operation (FLOP) refers to a single real multiplication or addition. However, due to the intended complex nature of the problem, every quantity in the algorithm is complex-valued, and this makes measuring `FLOPs` tricky. For instance, a single complex multiplication is composed of $z_1 \cdot z_2 = (a+jb)\,(c+jd) = ac-bd+j\,(ad+bc)$, thus requiring 4 real multiplications and 2 real additions (6 real `FLOPs`), while a complex addition requires 2 real `FLOPs`, one for each coordinate. The standard practice in numerical linear algebra is to use the ratio for a complex multiply-add as the unit, since that is the dominant operation in dense matrix algorithms, and also because hardware executes multiply-adds as fused operations anyways via FMA (Fused Multiply-Add) instructions. A complex multiply-add operation $c \leftarrow c+ab$ costs 1 complex multiplication and 1 complex addition, so 8 real operations in total per complex multiply-add.

Therefore, we adopt the convention used by `LAPACK` and `BLAS` in their own documentation: estimate the number of `FLOPs` given for each line as complex multiply-adds, where one complex multiply-add counts as a single unit. This matches the operation counts in performance benchmarks such as `LINPACK`. Under this convention, a dense $n \times n$ complex matrix-vector product (`zgemv`) costs $n^2$ complex multiply-adds, an LU factorization of an $n \times n$ complex matrix (`zgetrf`) costs $\frac{1}{3}n^3$ complex multiply-adds, and a triangular solve with one right-hand side (`zgetrs`, one RHS) costs $n^2$ complex multiply-adds.

To convert to (estimated) real `FLOPs`, multiply each figure by 8 (the real `FLOP` cost of one complex multiply-add, accounting for 4 real multiplications and 4 real additions: 2 to form the product and 2 to accumulate it).

It is natural that exploiting parameters and inner workings of operators in memory is paramount to reducing `FLOPs` number and the time required to complete the calculations, which we call "wall-clock time" to avoid ambiguity. For instance, suppose $A, B$ and $C$ are complex matrices and $\alpha$ is a complex scalar. At a first glance the equation $C \leftarrow AB + \alpha C$, quite common in numerical steps, would require two operations and an intermediate variable ($D \leftarrow AB$, $C \leftarrow D+C$); however, `LAPACK` and `BLAS` exploit cache-blocking and avoid temporaries, exposing functions like `zgemm` that allow specifying $\alpha, A, B$ and $\beta$ so that $C \leftarrow \alpha AB + \beta C$ is undertaken in an atomic, cache-efficient operation, reducing memory traffic and improving wall-clock time.

5.5. **Comparison with the realified method.** Remark 2 guarantees identical iterates, so the two methods can only differ in the cost of a step. For dense Wirtinger operators the comparison does not favor NRW. Under the convention above, (95) amounts to $\frac{8}{3}n^3$ complex multiply-adds, i.e., about $\frac{64}{3}n^3$ real `FLOPs`, whereas the LU factorization of the $2n \times 2n$ real Jacobian $J_R$ costs $\frac{1}{3}(2n)^3 = \frac{8}{3}n^3$ real multiply-adds, i.e., $\frac{16}{3}n^3$ real `FLOPs`:

| Method (dense) | Leading cost per iteration | Real `FLOPs` |
|---|---|---|
| NRW, Algorithm 1 | $\frac{8}{3}n^3$ complex multiply-adds | $\frac{64}{3}n^3$ |
| Realified NR, LU of $J_R \in \mathbb{R}^{2n\times 2n}$ | $\frac{8}{3}n^3$ real multiply-adds | $\frac{16}{3}n^3$ |

TABLE 1. Leading-order cost of one Newton step for dense Wirtinger operators.

Dense NRW is therefore about four times more expensive than realified Newton-Raphson for the same iterate. Nor does the Schur reduction save work relative to factorizing the $2n \times 2n$ complex augmented matrix $\mathcal{J}$ directly, which also costs $\frac{1}{3}(2n)^3 = \frac{8}{3}n^3$ complex multiply-adds. The reduction pays off only when the pivot block, $H_f$ or $W_f$ for the B-pivot, is cheap to invert: the $\frac{4}{3}n^3$ of line 3 and the $n^3$ of line 5 of Algorithm 1 then collapse to sparse operations, leaving a single $n \times n$ complex factorization of $\overline{\Sigma}$. This happens, for instance, when the pivot block is diagonal. When one

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of the operators is diagonal but vanishes on part of the domain, as $W_f$ does in the Gross-Pitaevskii application of Section 7, neither reduction is reliable and the augmented system is solved directly.

## 6. Application 1: Power flow in a single generator against infinite bus system

Power Flow is a natural setting for non-holomorphic rootfinding: in steady-state sinusoidal regimens, the state of an AC network is a vector of complex bus voltages $V \in \mathbb{C}^n$, but the nodal power balance involves terms of the form $V\,\overline{I}$, so the equations depend on both $V$ and $\overline{V}$, establishing the non-holomorphy. The established practice is to realify the problem in polar coordinates (that is, solve for the vectors of voltage amplitudes $|V_i|$, and voltage phases $\theta_i$) and apply the classical Newton-Raphson method [28].

Wirtinger calculus allows the problem to be solved directly in complex variables instead, with demonstrated computational gains in Newton power flow, approximate power flow, state estimation and quasi-dynamic simulation [13–15, 20, 21, 37, 39]. In those works convergence is assessed empirically or inherited from the realified method. This section makes the certificate of Theorems 5 and 7 explicit on a minimal example (Section 6.2), and shows that the iterates coincide with those of realified Newton-Raphson (Section 6.3), so that the analysis applies equally to the complex-variable implementations cited above.

### 6.1. The single generator against infinite bus system.

Figure 1 shows a simple circuit model of a SGIB system, the smallest instance of a Power Flow problem. The system is composed of a single generator $V_G$ that outputs a complex power $S_0$ to a large electrical grid $V_\infty$ through a transmission line of serial admittance $y$ and shunt admittance $y_{sh}$. The large grid is modeled as what is known as an "infinite bus", meaning an arbitrarily large system that maintains a constant voltage regardless of the power $S$ exchanged with the generator. Table 2 shows the parameters used.

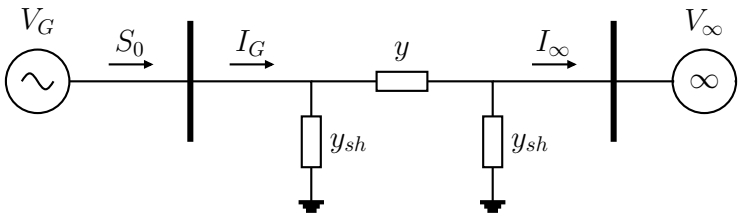



Figure 1. Single generator against infinite bus system example model.

| Parameter | Value |
|---|---|
| $y$ | $5.2420703 \times 10^{-5} - j0.27825418$ S |
| $y_{sh}$ | $5.2724656 \times 10^{-5} + j0.03861822$ S |
| $V_\infty$ | $100 + j0$ V |
| $S_0$ | $1000 + j100$ VA |

Table 2. Equivalent circuit parameters of the SGIB system of figure 1.

The power $S_0$ is fixed by the generator controllers, and the unknown is the terminal voltage $V_G$ that delivers it. By applying Kirchoff's Laws to the circuit and using $S_0 = V_G\overline{I_G}$, one obtains

$$V_\infty - \left(1 + \frac{y_{sh}}{y}\right) V_G - \frac{1}{y}\overline{\left(\frac{S_0}{V_G}\right)} = 0 \tag{98}$$

thus $V_G$ is a root of

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$$f(z) = V_\infty - \left(1 + \frac{y_{sh}}{y}\right) z - \frac{1}{y}\overline{\left(\frac{S_0}{z}\right)}. \tag{99}$$

and Figure 2 shows a domain-colored plot of $f$ using the parameters of table 2. By multiplying $f$ by $\overline{z}$ one obtains two real conics (two circles) whose intersection limits the number of roots to at most two; one can find the only two roots $z_1 \approx 98.79314254 - j35.96151994$ and $z_2 \approx 17.30421835 - j35.94111676$ analytically.

6.2. **Wirtinger formulation and Kantorovich certificate.** The Wirtinger operators allow us to work with $f(z)$ directly in the complex framework. We note that $f$ is of the form $f(z) = a+bz+c/\overline{z}$, thus manipulating the function is done in a simple manner:

$$\frac{\partial f}{\partial z} = -\left(1 + \frac{y_{sh}}{y}\right) \overbrace{\frac{\partial z}{\partial z}}^{=1} - \frac{1}{y}\overline{S_0}\ \overbrace{\frac{\partial}{\partial z}\frac{1}{\overline{z}}}^{=0} = -\left(1 + \frac{y_{sh}}{y}\right) \tag{100}$$

$$\frac{\partial f}{\partial \overline{z}} = -\left(1 + \frac{y_{sh}}{y}\right) \overbrace{\frac{\partial z}{\partial \overline{z}}}^{=0} - \frac{\overline{S_0}}{y}\ \overbrace{\frac{\partial}{\partial \overline{z}}\frac{1}{\overline{z}}}^{=-1/\overline{z}^2} = \frac{\overline{S_0}}{y\,\overline{z}^2} \tag{101}$$

Evaluating (100)–(101) with the parameters of Table 2 at the initial guess $z_0 = 100 - j40$ yields the Kantorovich inputs of Theorem 5. The Wirtinger operators are

$$A = -\left(1 + \frac{y_{sh}}{y}\right) \approx -0.86121 - j\,0.000216 \qquad \text{(constant in } z\text{)}, \tag{102}$$

$$B(z_0) = \frac{\overline{S_0}}{y\,\overline{z_0}^2} \approx 0.23614 + j\,0.20294. \tag{103}$$

The invertibility criterion $|A|^2 - |B(z_0)|^2 \approx 0.645 \neq 0$ is satisfied. For the scalar case, Lemma 3 gives the closed form $\left\|[L_{(f,z_0)}]^{-1}\right\| = 1/(|A| - |B(z_0)|)$, so

$$\beta = \frac{1}{|A| - |B(z_0)|} \approx \frac{1}{0.86121 - 0.31136} \approx 1.819. \tag{104}$$

Reading the first NRW step from Table 3 gives $\eta = |z_1 - z_0| \approx 4.235$. Since $A$ is constant and $B(z) = \overline{S_0}/(y\overline{z}^2)$, the Lipschitz constant of $z \mapsto L_{(f,z)}$ on a ball around $z_0$ is governed by $|B'(z)| = 2|\overline{S_0}/y|/|z|^3$. Hypothesis (K3) requires this bound over the whole ball $\overline{B}(z_0, t_*)$, not only at $z_0$, and $|B'|$ grows as $|z|$ decreases; hence $L$ is taken as the supremum over the ball, $L = 2|\kappa|/(|z_0|-t_*)^3$ with $t_* = (1 - \sqrt{1-2h})/(\beta L)$, solved self-consistently. This yields $L \approx 6.54 \times 10^{-3}$ (against $5.78 \times 10^{-3}$ at $z_0$ alone), giving

$$h = \beta L \eta \approx 1.819 \times 6.54 \times 10^{-3} \times 4.235 \approx 0.050 < \tfrac{1}{2}, \tag{105}$$

confirming convergence, with convergence radius $t_* \approx 4.35$.

For the second initial guess $z_0 = 15 - j37$, $A$ is unchanged. At this point $\overline{z_0}^2 = -1144 + j1110$, so

$$B\,(z_0) = \frac{\overline{S_0}}{y\ \overline{z_0}^2} \approx 1.4079 - j\,1.7754. \tag{106}$$

Here $|B(z_0)| \approx 2.266 > |A| \approx 0.861$, so $B$ dominates and the B-pivot formula (43) is the natural one; the A-pivot gives the same step. The operator norm formula gives $\beta = 1/(|B(z_0)| - |A|)$:

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$$\beta = \frac{1}{|B(z_0)| - |A|} \approx \frac{1}{2.266 - 0.861} \approx 0.712. \tag{107}$$

The two initial guesses lie on opposite sides of a curve with a physical meaning. For $n = 1$ the realified Jacobian has $\det J_R = |A|^2 - |B(z)|^2$, and $|B(z)| = |S_0|/(|y|\,|z|^2)$, so the Jacobian is singular exactly on the circle $|z| = \sqrt{|S_0|/(|y|\,|A|)} \approx 64.8$. The normal operating point $z_1$ lies outside it, where $A$ dominates ($|B(z_1)| \approx 0.327$), and the low-voltage root $z_2$ lies inside, where $B$ dominates ($|B(z_2)| \approx 2.270$). As $S_0$ is scaled up, the two roots approach the circle from opposite sides and coalesce on it at about $1.46\,S_0$. The A-dominant and B-dominant regions therefore surround the two branches of the nose curve of voltage stability [40], and the singular circle that separates them is where the operating point ceases to exist. At the coalescence loading $\lambda_c \approx 1.461\,S_0$ the singular circle radius has itself grown to $\sqrt{|S_0|\lambda_c/(|y|\,|A|)} = 64.760 \times \sqrt{1.461} \approx 78.3$, the same as the coalescence locus $|z_{\text{fold}}| \approx 78.3$: the two roots merge exactly on the singular circle. The voltage-collapse point is therefore a root-fold coincidence in codimension 2, connecting the transfer limit of the line to the geometry of the Wirtinger operators.

Reading the first NRW step from Table 3 gives $\eta = |z_1 - z_0| \approx 2.611$. As for the first guess, the Lipschitz constant is taken as the supremum over the convergence ball $B(z_0, t_*)$ rather than at $z_0$ alone, which matters more here since $|z_0| = \sqrt{1594} \approx 39.92$ is smaller: solving the fixed-point equation $L = 2|\kappa|/(|z_0| - t_*)^3$ with $t_* = (1 - \sqrt{1 - 2h})/(\beta L)$ self-consistently yields $L \approx 0.1448$ and $t_* \approx 3.11$, giving

$$h = \beta L \eta \approx 0.712 \times 0.1448 \times 2.611 \approx 0.269 < \tfrac{1}{2}. \tag{108}$$

The certificate holds comfortably within the Kantorovich bound. The self-consistent determination of $L$ over the ball is, however, the weakest step of the certificate, since $\beta$ and $L$ are bounded separately. For this scalar problem the affine-covariant Theorem 6 does not help: as shown in Section 4.2, $\omega = \beta L$ exactly. Its advantage appears in the multidimensional problem of Section 7.

Figure 2 shows a domain-colored plot of (99); in that plot, both roots are evident, as well as the conjugate pole at the origin. One immediately notices that, near the roots, the level curves become ellipses that show geometrically the non-holomorphicity of $f(z)$ since a holomorphic function would plot level curves into circles. Calculating the NRW step matrices $\Gamma, \Sigma$ using (100) and (101):

$$\Gamma = \left[\frac{1}{\overline{y}}\frac{S_0}{(z)^2}\right]\left(1 + \frac{y_{sh}}{y}\right)^{-1} \tag{109}$$

$$\Sigma = -\left(1 + \frac{\overline{y_{sh}}}{\overline{y}}\right) + \left(\frac{|S_0|}{|z|^2\,|y|}\right)^2 \left(1 + \frac{y_{sh}}{y}\right)^{-1} \tag{110}$$

Using these matrices and given proper initial guesses, table 3 shows the iteration progression of the method for both initial guesses.

We notice that the iterations from the first initial guess have a clear quadratic convergence, from the progression of the iterations, as well as the fact that both iterations converge to the roots to machine precision (order $10^{-16}$). This is a consequence of the low $h$ constant in (105). For the second initial guess, a modestly larger $h \approx 0.269$ (equation (108)) slows convergence on the first step, after which the characteristic quadratic regime is clearly visible from the second step onward.

6.3. **Equivalence with the realified Newton-Raphson.** Remark 2 establishes in general that NRW and realified Newton-Raphson produce identical iterates. We verify this concretely for the SGIB function (99) by working entirely from $f$ and showing every step analytically.

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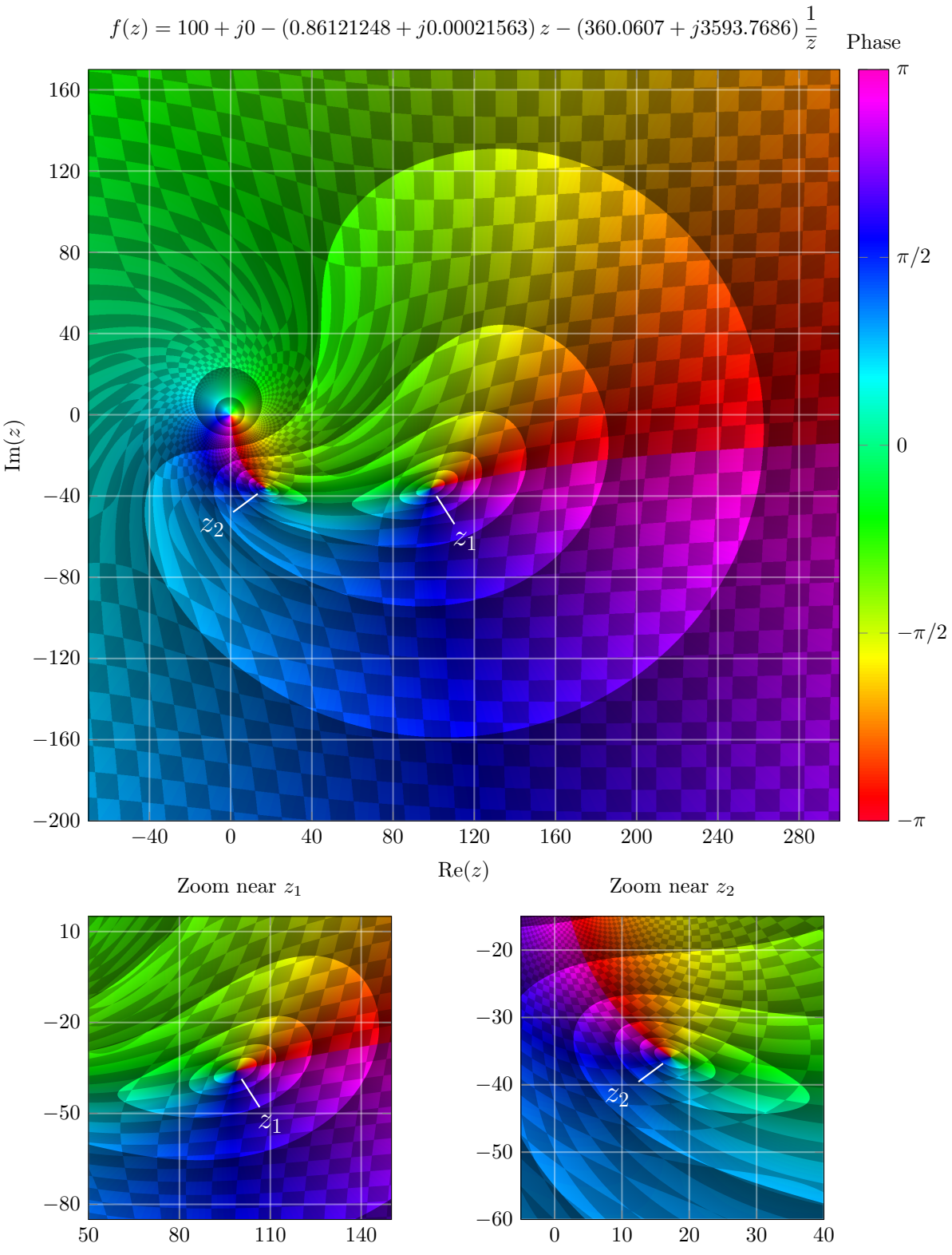



FIGURE 2. Domain-colored complex plot of the function (99) using the parameters of table 2. Color gradient shows phase variation, and shaded curves are level curves in powers of two. The checkerboard pattern shows estimation of local distortion.

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| $k$ | $z_k$ | $e_k = \lvert z_k - z^* \rvert$ | $e_{k+1}/e_k^2$ |
|---|---|---|---|
| 0 | $100.0000000000000 - j40.00000000000000$ | 4.2150 | $4.724 \times 10^{-3}$ |
| 1 | $98.70936857483341 - j35.96653482711481$ | $8.392 \times 10^{-2}$ | $5.813 \times 10^{-3}$ |
| 2 | $98.79318101770922 - j35.96150595257353$ | $4.094 \times 10^{-5}$ | $4.87 \times 10^{-3}$ |
| 3 | $98.79314253845800 - j35.96151994098160$ | $8.2 \times 10^{-12}$ | — |
| 4 | $98.79314253844986 - j35.96151994098201$ | — | — |

| $k$ | $z_k$ | $e_k = \lvert z_k - z^* \rvert$ | $e_{k+1}/e_k^2$ |
|---|---|---|---|
| 0 | $15.00000000000000 - j37.00000000000000$ | 2.5359 | $3.292 \times 10^{-2}$ |
| 1 | $17.44907565890279 - j36.09548794780686$ | $2.117 \times 10^{-1}$ | $4.039 \times 10^{-2}$ |
| 2 | $17.30258249398777 - j35.94034247464968$ | $1.810 \times 10^{-3}$ | $3.445 \times 10^{-2}$ |
| 3 | $17.30421823680668 - j35.94111675594021$ | $1.128 \times 10^{-7}$ | — |
| 4 | $17.30421834933542 - j35.94111676417847$ | — | — |

TABLE 3. NRW iterations for (99). $e_k = \lvert z_k - z^* \rvert$ is the absolute error to the exact root; $e_{k+1}/e_k^2$ is the observed quadratic convergence ratio. The theoretical bound from Theorem 7 gives $C = \beta^* L$: for the first run $C_1 \approx 1.163 \times 10^{-2}$ and for the second $C_2 \approx 8.08 \times 10^{-2}$; all observed ratios lie strictly below their bound. The root $z^*$ is computed in double precision, which determines it only up to about $10^{-13}$; errors are given to the digits this allows, and entries marked "—" are below that level.

Let $z = z_R + jz_I$ with $z_R, z_I \in \mathbb{R}$, and recall $\kappa := \overline{S_0}/y = \kappa_R + j\kappa_I$, $A = A_R + jA_I$. The two terms of $f(z) = V_\infty + Az - \kappa/\overline{z}$ expand as follows. For the linear term,

$$Az = (A_R + jA_I)(z_R + jz_I) = (A_R z_R - A_I z_I) + j(A_I z_R + A_R z_I). \tag{111}$$

For the conjugate-reciprocal term, rationalize $\overline{z} = z_R - jz_I$ by multiplying numerator and denominator by $z_R + jz_I$:

$$\frac{\kappa}{\overline{z}} = \frac{(\kappa_R + j\kappa_I)(z_R + jz_I)}{|z|^2} = \frac{(\kappa_R z_R - \kappa_I z_I) + j(\kappa_I z_R + \kappa_R z_I)}{|z|^2}. \tag{112}$$

Separating $f = u + jv$ with $V_R = \mathrm{Re}(V_\infty) = 100\,\mathrm{V}$ and $\mathrm{Im}(V_\infty) = 0$:

$$u(z_R, z_I) = V_R + A_R z_R - A_I z_I - \frac{\kappa_R z_R - \kappa_I z_I}{|z|^2}, \tag{113}$$

$$v(z_R, z_I) = A_I z_R + A_R z_I - \frac{\kappa_I z_R + \kappa_R z_I}{|z|^2}. \tag{114}$$

The realified Newton-Raphson step at $z_k$ solves the $2 \times 2$ real system

$$J_R(z_k) \begin{bmatrix} \Delta z_R \\ \Delta z_I \end{bmatrix} = - \begin{bmatrix} u(z_k) \\ v(z_k) \end{bmatrix}, \qquad z_{k+1} = z_k + \Delta z_R + j\Delta z_I, \tag{115}$$

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where $J_R = \partial(u,v)/\partial(z_R, z_I)$ is the real Jacobian. Treating $\overline{z} = z_R - jz_I$ as a function of the two real variables gives $\partial\overline{z}/\partial z_R = 1$ and $\partial\overline{z}/\partial z_I = -j$, so the partial derivatives of $\kappa/\overline{z}$ are

$$\frac{\partial}{\partial z_R}\frac{\kappa}{\overline{z}} = -\frac{\kappa}{\overline{z}^2} = -B(z), \qquad \frac{\partial}{\partial z_I}\frac{\kappa}{\overline{z}} = \frac{j\kappa}{\overline{z}^2} = jB(z). \tag{116}$$

Differentiating (113)–(114) and using (116):

$$\frac{\partial u}{\partial z_R} = A_R + \mathrm{Re}(B(z)) = A_R + B_R(z), \tag{117}$$

$$\frac{\partial u}{\partial z_I} = -A_I + \mathrm{Im}(B(z)) = B_I(z) - A_I, \tag{118}$$

$$\frac{\partial v}{\partial z_R} = A_I + \mathrm{Im}(B(z)) = A_I + B_I(z), \tag{119}$$

$$\frac{\partial v}{\partial z_I} = A_R - \mathrm{Re}(B(z)) = A_R - B_R(z). \tag{120}$$

Assembling these four entries gives the Jacobian:

$$J_R(z) = \begin{bmatrix} A_R + B_R(z) & B_I(z) - A_I \\ A_I + B_I(z) & A_R - B_R(z) \end{bmatrix}. \tag{121}$$

Substituting the SGIB parameters $A \approx -0.86121 - j0.000216$ and evaluating $B(z_0) = \kappa/\overline{z_0}^2$ at the first initial guess $z_0 = 100 - j40$, so $\overline{z_0}^2 = (100 + j40)^2 = 8400 + j8000$ and $B(z_0) \approx 0.23614 + j0.20294$, gives:

$$J_R(z_0) \approx \begin{bmatrix} -0.6251 & 0.2032 \\ 0.2027 & -1.0973 \end{bmatrix}. \tag{122}$$

To confirm agreement with the NRW step, write $\Delta z = \Delta z_R + j\Delta z_I$ in the widely-linear system $A\,\Delta z + B(z_k)\,\overline{\Delta z} = -f(z_k)$ and separate real and imaginary parts:

$$\underbrace{\begin{bmatrix} A_R + B_R & B_I - A_I \\ A_I + B_I & A_R - B_R \end{bmatrix}}_{= J_R(z_k)} \begin{bmatrix} \Delta z_R \\ \Delta z_I \end{bmatrix} = -\begin{bmatrix} u(z_k) \\ v(z_k) \end{bmatrix}. \tag{123}$$

The coefficient matrix is precisely $J_R(z_k)$ from (121): the two linear systems (115) and (123) are identical, confirming that both methods produce the same iterates.

To verify numerically, we carry out the first step at each initial guess. At $z_0 = 100 - j40$, with $|z_0|^2 = 11600$, substituting the SGIB parameters into (113)–(114) gives

$$u(z_0) = 100 + (-0.8612)(100) - (-0.000216)(-40) - \frac{(360.06)(100) - (3593.8)(-40)}{11600} \approx -1.626,$$

$$v(z_0) = (-0.000216)(100) + (-0.8612)(-40) - \frac{(3593.8)(100) + (360.06)(-40)}{11600} \approx 4.688.$$

With $J_R(z_0)$ from (122), the realified Newton-Raphson step solves

$$\begin{bmatrix} -0.6251 & 0.2032 \\ 0.2027 & -1.0973 \end{bmatrix} \begin{bmatrix} \Delta z_R \\ \Delta z_I \end{bmatrix} = \begin{bmatrix} 1.626 \\ -4.688 \end{bmatrix}, \tag{124}$$

giving $\Delta z_R \approx -1.291$, $\Delta z_I \approx 4.033$, and $z_1 \approx 98.709 - j35.967$.

At $z_0 = 15 - j37$, with $|z_0|^2 = 1594$ and $B(z_0) \approx 1.4079 - j1.7754$ (from Section 6.2), equations (118)–(120) give

$$J_R(z_0) \approx \begin{bmatrix} 0.5467 & -1.7752 \\ -1.7756 & -2.2691 \end{bmatrix}. \tag{125}$$

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Evaluating the functions:

$$u(z_0) = 100 + (-0.8612)(15) - (-0.000216)(-37) - \frac{(360.06)(15) - (3593.8)(-37)}{1594} \approx 0.267,$$
$$v(z_0) = (-0.000216)(15) + (-0.8612)(-37) - \frac{(3593.8)(15) + (360.06)(-37)}{1594} \approx 6.401.$$

Solving $J_R(z_0)\,[\Delta z_R, \Delta z_I]^\top = -\,[0.267, 6.401]^\top$ gives $\Delta z_R \approx 2.449$, $\Delta z_I \approx 0.905$, so $z_1 \approx 17.449 - j36.095$.

Both $z_1$ values coincide exactly with the NRW step applied to the same initial guess. Continuing both methods from $z_0 = 15 - j37$ yields Table 4, in which each $z_k$ is identical to the corresponding entry in the second subtable of Table 3.

| $k$ | $z_{R,k}$ | $z_{I,k}$ | $u(z_k)$ | $v(z_k)$ |
|---|---|---|---|---|
| 0 | 15.00000000000000 | −37.00000000000000 | $2.668 \times 10^{-1}$ | $6.401 \times 10^{0}$ |
| 1 | 17.44907565890279 | −36.09548794780686 | $3.528 \times 10^{-1}$ | $1.547 \times 10^{-1}$ |
| 2 | 17.30258249398777 | −35.94034247464968 | $-2.476 \times 10^{-3}$ | $6.694 \times 10^{-4}$ |
| 3 | 17.30421823680668 | −35.94111675594021 | $-9.894 \times 10^{-8}$ | $1.579 \times 10^{-7}$ |
| 4 | 17.30421834933542 | −35.94111676417847 | — | — |

TABLE 4. Realified Newton-Raphson iterates from $z_0 = 15 - j37$ for the SGIB function (99). Columns $z_{R,k}$ and $z_{I,k}$ are the real and imaginary parts of $z_k$; $u(z_k)$ and $v(z_k)$ are the residual components at each step. These $z_k$ values coincide with the second subtable of Table 3, confirming that the realified NR and NRW methods produce the same sequence. At $k = 4$ the residuals are below the rounding level of $f$ in double precision ($\approx 5 \times 10^{-14}$) and are marked "—".

The realified path required: (i) expanding $f$ into $u$ and $v$ via (111)–(114), (ii) computing four partial derivatives via (118)–(120), and (iii) assembling $J_R$. The NRW path required only: evaluating $A = \partial f/\partial z$ and $B(z_k) = \partial f/\partial \overline{z}$ from (100)–(101) and reading off $J_R$ directly from (121) — two complex derivatives instead of four real ones. The same advantage extends to the convergence analysis: the Kantorovich constants $\beta = 1/(|A| - |B(z_0)|)$ and $L = 2|\kappa|/(|z_0| - t_*)^3$ were obtained from scalar moduli of the Wirtinger operators (see (105)–(108)), whereas computing equivalent constants from the realified $J_R$ would require the operator norm $\left\|J_R^{-1}\right\|$ and the Lipschitz constant of $z \mapsto J_R(z)$ in the real $2 \times 2$ setting — more involved calculations that scale to $2n \times 2n$ for an $n$-dimensional problem.

## 7. Application 2: Vortex states of the Gross-Pitaevskii equation

The single-generator example of Section 6 is scalar, and the Kantorovich certificate closes comfortably there. The second application is chosen to show the opposite situation. It is a large, sparse problem from a different field in which the certificate fails by orders of magnitude although Newton's method converges quadratically. This illustrates that the Kantorovich condition is sufficient but not necessary, and it motivates the affine-invariant refinement mentioned at the end of Section 6.2.

The Gross-Pitaevskii (GP) equation models a Bose-Einstein condensate: a gas of atoms cooled so close to absolute zero that they all occupy the same quantum state and behave collectively as a single object [4, 10]. The state of the condensate is described by a complex-valued function of position, the wave function $u : \mathbb{R}^2 \to \mathbb{C}$: $|u(x)|^2$ is the density of atoms at the point $x$, and $\arg u(x)$ is a phase. We consider *stationary* states, which change in time only by a uniform phase factor, so

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that $u$ depends on space alone and the problem is a nonlinear partial differential equation in the plane. The atoms are held near the origin by a trap, modeled by a potential $V$ that grows with the distance from the origin. In units adapted to the trap, the stationary states solve

$$-\tfrac{1}{2}\Delta u + V(x)\,u + g\,|u|^2\,u - \mu u = 0, \qquad V(x) = \tfrac{1}{2}\,|x|^2\,, \quad x \in \mathbb{R}^2, \tag{126}$$

Each term has an elementary meaning. The term $-\frac{1}{2}\Delta u$, with $\Delta$ the Laplacian, is the kinetic energy and penalizes rapid spatial variation of $u$. The term $V(x)\,u$ confines the atoms. The term $g\,|u|^2\,u$, with $g > 0$, models the repulsion between atoms, which is stronger where the density is higher. The real parameter $\mu$, the chemical potential, is given and in effect fixes the number of atoms. Without the cubic term, (126) is a linear eigenvalue problem for the harmonic oscillator; the cubic term makes it nonlinear, and it is the only term in which $\overline{u}$ appears.

Two length scales follow from (126). Where the atoms are dense the Laplacian term is negligible and $|u|^2 \approx (\mu - V)/g$, a profile that vanishes at the edge of the condensate, the Thomas-Fermi radius $r_{\mathrm{TF}} = \sqrt{2\mu}$. Where $u$ must vary rapidly, the Laplacian and the interaction balance over the healing length $\xi = 1/\sqrt{2\mu}$. We seek a *vortex*, a whirlpool in the condensate [16, 27]: a solution that vanishes at the origin and whose phase increases by $2\pi$ around it, so that in polar coordinates $(r, \varphi)$, $u \approx \rho(r)e^{j\varphi}$, with $\rho(0) = 0$ and $\rho$ growing over a distance of order $\xi$.

This problem benefits from the NRW formulation for three reasons. First, ground states can be taken real and positive, and Newton-type methods for them are well developed [19], whereas a vortex is genuinely complex: the cubic term $|u|^2\,u = u^2\overline{u}$ makes (126) non-holomorphic, so the classical Newton method in $\mathbb{C}^n$ does not apply. Second, written in $u$ and $\overline{u}$ the nonlinearity is a monomial, so the Wirtinger operators follow by inspection (Section 7.1): $H_f$ is the discretized linear operator plus $2g\,\mathrm{diag}(|u|^2)$, and all the coupling between $u$ and $\overline{u}$ sits in the diagonal $W_f = g\,\mathrm{diag}(u^2)$. Under a global phase rotation $u \mapsto e^{j\theta}u$, $H_f$ is unchanged and $W_f$ is multiplied by $e^{2j\theta}$, which Section 7.2 uses to identify the degenerate direction $ju_*$ at every root and to pin the phase without destroying this structure. Realifying with $u = a + jb$ instead gives two coupled real equations with cubic terms $(a^2+b^2)a$ and $(a^2+b^2)b$, and a Jacobian whose blocks mix $a$ and $b$ and hide this invariance. Third, in a non-rotating trap the vortex is an excited stationary state rather than an energy minimizer, so the imaginary-time and gradient-flow methods used for ground states do not target it, while Newton's method converges to stationary states regardless of their stability.

7.1. **Discretization and Wirtinger operators.** We discretize (126) on the square $[-a, a]^2$ with homogeneous Dirichlet conditions, using $N$ interior points per direction and the five-point Laplacian $\Delta_h$. With $A_h = -\frac{1}{2}\Delta_h + \mathrm{diag}(V)$, a real symmetric sparse matrix, the unknown $u \in \mathbb{C}^n$, $n = N^2$, solves

$$f(u) = (A_h - \mu I)\,u + g\,\mathrm{diag}\left(|u|^2\right)u = 0. \tag{127}$$

Since $|u_i|^2\,u_i = u_i^2\overline{u_i}$, the Wirtinger operators follow by inspection:

$$H_f(u) = A_h - \mu I + 2g\,\mathrm{diag}\left(|u|^2\right), \qquad W_f(u) = g\,\mathrm{diag}\left(u^2\right). \tag{128}$$

$H_f$ is real, symmetric and has the five-point sparsity pattern, and $W_f$ is diagonal. A diagonal $W_f$ would suggest the B-pivot of Lemma 3, that is, Newton's method with $\overline{u}$ as the unknown. However, $W_f$ vanishes at the vortex core and is negligible outside the condensate, where $u$ decays. $H_f$ contains $-\mu I$ and is indefinite. Neither pivot is therefore uniformly well conditioned, and we solve the augmented system $\mathcal{J}[h; \overline{h}]^{\intercal} = -[f; \overline{f}]^{\intercal}$ directly, as provided for in Lemma 3; this is the pure Newton step of Section 5.3 applied to the augmented system. Ordering the unknowns as $(u_1, \overline{u_1}, \ldots, u_n, \overline{u_n})$, the five-point pattern of $H_f$ and the diagonal $W_f$ make $\mathcal{J}$ a band matrix with $2N$ subdiagonals and $2N$ superdiagonals. The step is therefore computed with the banded LU factorization of LAPACK (`zgbtrf` and `zgbtrs`), in $O(nN^2)$ operations and $O(nN)$ storage instead of the $O(n^3)$ operations and $O(n^2)$ storage of a dense factorization of $\mathcal{J}$. The same factors are reused

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for the simplified corrections of the affine-covariant estimates (73) and for the inverse iteration that computes $\sigma_{\min}(\mathcal{J})$. The choice affects only how the step is computed: the iterates are those of the NRW method whichever pivot or solver is used.

7.2. **Phase invariance and phase pinning.** Equation (127) is invariant under a global phase change: $f(e^{j\theta}u) = e^{j\theta}f(u)$ for every $\theta \in \mathbb{R}$. Roots therefore come in circles $\{e^{j\theta}u_*\}$ and are never isolated. Differentiating at $\theta = 0$ shows that the widely-linear derivative at any root annihilates the tangent direction $ju_*$:

$$\begin{aligned} L_{(f,u_*)}(ju_*) &= jH_f u_* - jW_f\overline{u_*} = j\left[(A_h - \mu I)u_* + 2g\,|u_*|^2 u_* - g\,|u_*|^2 u_*\right] \\ &= jf(u_*) = 0. \end{aligned}$$

Hence $\beta = \infty$ at every root, and neither Theorem 5 nor Theorem 7 applies as stated. The phase must be fixed. We pin it at a grid point $k$ where the condensate is dense, through the modified map

$$\tilde{f}(u) = f(u) + j\gamma\, e_k\, \mathrm{Im}(u_k), \qquad \gamma > 0, \tag{129}$$

where $e_k$ is the $k$-th canonical vector.

**Proposition 8** (Phase pinning)**.** *Let $u \in \mathbb{C}^n$ with $\mathrm{Re}(u_k) \neq 0$. Then $\tilde{f}(u) = 0$ if and only if $f(u) = 0$ and $\mathrm{Im}(u_k) = 0$. The Wirtinger operators of $\tilde{f}$ are*

$$H_{\tilde{f}} = H_f + \tfrac{\gamma}{2} e_k e_k^{\mathsf{T}}, \qquad W_{\tilde{f}} = W_f - \tfrac{\gamma}{2} e_k e_k^{\mathsf{T}},$$

*so pinning changes one diagonal entry of each operator and preserves their sparsity and their Lipschitz constant. Moreover, $L_{(\tilde{f},u_*)}(ju_*) = j\gamma\,\mathrm{Re}(u_{*,k})\,e_k \neq 0$ at every root $u_*$ of $\tilde{f}$, so the phase direction is no longer in the kernel.*

*Proof.* Since $A_h$ is real symmetric, $u^{\mathsf{H}} f(u) = u^{\mathsf{H}}(A_h - \mu I)u + g\sum_i |u_i|^4$ is real for every $u$. If $\tilde{f}(u) = 0$, then $f(u) = -j\gamma s\, e_k$ with $s = \mathrm{Im}(u_k)$, and $u^{\mathsf{H}} f(u) = -j\gamma s\,\overline{u_k}$. This is real only if $\gamma s\,\mathrm{Re}(u_k) = 0$, so $s = 0$ and $f(u) = 0$. The converse is immediate. Writing $j\,\mathrm{Im}(u_k) = \frac{1}{2}(u_k - \overline{u_k})$ gives the Wirtinger operators. The last claim follows from $L_{(f,u_*)}(ju_*) = 0$ and $\mathrm{Im}(ju_{*,k}) = \mathrm{Re}(u_{*,k})$. □

The Lipschitz constant of $u \mapsto L_{(\tilde{f},u)}$ is not global, since $f$ is cubic, but it is explicit on a ball. By (128), $H_f(u) - H_f(v) = 2g\,\mathrm{diag}(|u|^2 - |v|^2)$ and $W_f(u) - W_f(v) = g\,\mathrm{diag}(u^2 - v^2)$. If $|u_i|, |v_i| \leq M$ for all $i$, then both $\left||u_i|^2 - |v_i|^2\right|$ and $\left|u_i^2 - v_i^2\right|$ are at most $2M\,|u_i - v_i| \leq 2M\,|u - v|$. The bound of Definition 3 then gives, on $\overline{B}(u_0, t)$,

$$\left\|L_{(\tilde{f},u)} - L_{(\tilde{f},v)}\right\| \leq L(t)\,|u - v|\,, \qquad L(t) = 6g\,(\|u_0\|_\infty + t)\,. \tag{130}$$

7.3. **A vortex in a harmonic trap.** We take $a = 5$, $N = 64$ (so $n = 4096$ and grid spacing 0.154), $g = 1$ and $\mu = 5$, for which the Thomas-Fermi radius is $r_{\mathrm{TF}} = \sqrt{2\mu} \approx 3.16$. The phase is pinned at the grid point closest to $(r_{\mathrm{TF}}/2, 0)$, with $\gamma = 100$. The initial guess is the Thomas-Fermi profile with a unit-charge vortex at the origin,

$$u_0 = \sqrt{\max\{\mu - V, 0\}/g}\,\tanh\!\left(\frac{r}{\sqrt{2}\,\xi}\right) e^{j\varphi}, \qquad \xi = \frac{1}{\sqrt{2\mu}},$$

rotated by a constant phase so that $\mathrm{Im}(u_{0,k}) = 0$.

The NRW iteration converges in six steps to the vortex state $u_*$ of Figure 3, with $\left|\tilde{f}(u_*)\right| \approx 5 \times 10^{-13}$ and $\max_i |u_{*,i}| = 1.916$. Without pinning, the smallest singular value of $\mathcal{J}$ at $u_*$ is $2 \times 10^{-15}$, the numerical trace of the phase invariance. With pinning it is $3.6 \times 10^{-2}$. The operators have $\mathrm{nnz}(H_{\tilde{f}}) = 20224$ and $\mathrm{nnz}(W_{\tilde{f}}) = 4096$, against 48640 real nonzeros in the $8192 \times 8192$ realified Jacobian. Since $H_{\tilde{f}}$ is real (Proposition 8), the two off-diagonal blocks of $J_R$ reduce to $B_I$ alone, which is diagonal; so $\mathrm{nnz}(J_R) = 2 \times 20224 + 2 \times 4096 = 48640$, still twice the complex count

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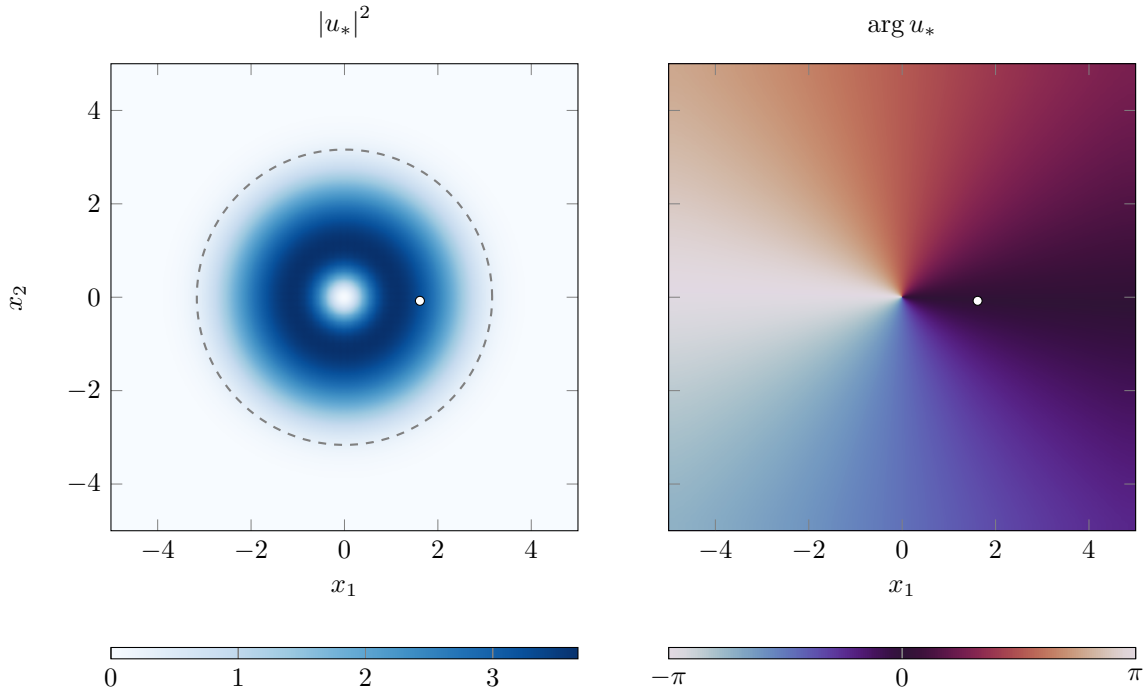



FIGURE 3. Vortex state $u_*$ of the discretized Gross-Pitaevskii equation (127) ($N = 64$, $g = 1$, $\mu = 5$). *Left*: density $|u_*|^2$, vanishing at the vortex core; the dashed circle is the Thomas-Fermi radius $r_{\mathrm{TF}} = \sqrt{2\mu}$. *Right*: phase $\arg u_*$, winding once by $2\pi$ around the core. The white dot marks the grid point $k$ at which the phase is pinned, where $\mathrm{Im}(u_{*,k}) = 0$.

20224+4096. Table 5 shows the iterations: convergence is quadratic, and the NRW iterates coincide with those of the realified Newton-Raphson method to within $10^{-12}$, as required by Remark 2.

| $k$ | $e_k = \lvert u_k - u_* \rvert$ | $e_{k+1}/e_k^2$ | $\lvert f(u_k) \rvert$ | $\lvert u_k^{\mathrm{NRW}} - u_k^{\mathrm{NR}} \rvert$ |
|---|---|---|---|---|
| 0 | $1.007 \times 10^{1}$ | $9.68 \times 10^{-3}$ | $1.888 \times 10^{2}$ | 0 |
| 1 | $9.826 \times 10^{-1}$ | $2.30 \times 10^{-2}$ | 3.752 | $4.7 \times 10^{-14}$ |
| 2 | $2.225 \times 10^{-2}$ | $2.61 \times 10^{-1}$ | $7.121 \times 10^{-2}$ | $1.3 \times 10^{-13}$ |
| 3 | $1.294 \times 10^{-4}$ | $1.27 \times 10^{-1}$ | $1.154 \times 10^{-4}$ | $1.8 \times 10^{-13}$ |
| 4 | $2.119 \times 10^{-9}$ | — | $2.367 \times 10^{-9}$ | $3.4 \times 10^{-13}$ |
| 5 | $5.6 \times 10^{-14}$ | — | $5.4 \times 10^{-13}$ | $2.6 \times 10^{-13}$ |

TABLE 5. NRW iterations for the GP vortex from the Thomas-Fermi initial guess $u_0$. The last column is the distance between the NRW iterate and the realified Newton-Raphson iterate from the same $u_0$. Entries marked "—" are at the level of rounding errors.

7.4. **Sufficient, not necessary: the case for affine invariance.** Table 6 evaluates the certificate of Theorem 5 at $u_0$ and at random perturbations of $u_*$ of decreasing size, with $\beta = 1/\sigma_{\min}(\mathcal{J})$. To favor the certificate, the table reports $h$ with $L$ evaluated at the initial guess, i.e., $L(0)$ in (130),

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| Initial guess | $\beta$ | $\eta$ | $L$ | $h$ | Certified |
|---|---|---|---|---|---|
| Thomas-Fermi $u_0$ | 2.29 | 10.27 | 12.44 | 292.7 | no |
| $u_*$ + perturbation of size $10^{-1}$ | 27.83 | $1.127 \times 10^{-1}$ | 11.51 | 36.1 | no |
| $u_*$ + perturbation of size $10^{-2}$ | 27.86 | $1.000 \times 10^{-2}$ | 11.50 | 3.20 | no |
| $u_*$ + perturbation of size $10^{-3}$ | 27.88 | $1.037 \times 10^{-3}$ | 11.50 | 0.333 | yes |
| $u_*$ + perturbation of size $10^{-4}$ | 27.88 | $1.028 \times 10^{-4}$ | 11.50 | 0.033 | yes |

TABLE 6. Kantorovich certificate for the GP vortex. In the certified rows, $L$ is the self-consistent ball constant $L(t_*)$ of (130).

which is a lower bound for the constant required over the ball. When this optimistic $h$ already exceeds $\frac{1}{2}$, the certificate fails for every ball.

From the Thomas-Fermi guess, $h \approx 293$, almost three orders of magnitude above the threshold, yet Table 5 shows that the iteration converges quadratically from that same point. The certificate only closes within about $10^{-3}$ of the root. Kantorovich's condition is therefore sufficient for convergence but far from necessary here, and the reason is structural. The theorem bounds $\left\|[L_{(\tilde{f},u_0)}]^{-1}\right\|$ and the variation of $L_{(\tilde{f},u)}$ separately, and the two bounds are attained in different directions. $\beta$ is governed by the soft modes of the vortex: displacements of the core, which cost little energy in a non-rotating trap, account for the small singular values of $\mathcal{J}$ near $u_*$. $L$ is governed by the dense region of the condensate, where $|u|$ is largest. The product $\beta L$ therefore combines a worst case of each kind, even though the Newton correction never sees both at once.

The affine-covariant Theorem 6 avoids this by bounding the composition $L_0^{-1} \circ (L_{(\tilde{f},u)} - L_{(\tilde{f},v)})$ directly. From the Thomas-Fermi guess, with $\eta = 10.27$, it would certify convergence as soon as $\omega \leq 1/(2\eta) = 4.87 \times 10^{-2}$. The constant $\omega$ is not available in closed form for this problem, but the estimates (73) are computed along the iteration at the cost of one additional solve per step with the matrix already factorized. At the first step we obtain $\Theta_0 = 0.090$ and $\hat{\omega}_0 = 1.75 \times 10^{-2}$, so $\hat{\omega}_0 \eta = 0.18 < \frac{1}{2}$. At the same point, $\beta L = 28.5$. The subsequent contraction factors are $\Theta_1 = 2.2 \times 10^{-2}$ and $\Theta_2 = 5.6 \times 10^{-3}$.

Since $\hat{\omega}_0$ is a lower bound for $\omega$, these numbers do not certify convergence. They do show that the affine-covariant constant can be smaller than $\beta L$ by three orders of magnitude here, which is exactly the gap between the observed convergence and the prediction of Theorem 5. They are also consistent with the threshold $\omega \leq 4.87 \times 10^{-2}$ required by Theorem 6. What is missing is a rigorous, computable upper bound on $\omega$ in terms of the Wirtinger operators that exploits the structure of the problem. We leave it for future work.

## 8. Conclusion

This paper developed the convergence theory of the Newton-Raphson-Wirtinger (NRW) method natively within the complex domain, treating $\mathbb{C}^n$ as a real-linear algebra rather than invoking the isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The distinction is not merely formal: realification obscures the algebraic structure of $f$ — a diagonal $\partial f/\partial z$, for instance, is mixed into every block of the $2n \times 2n$ real Jacobian — whereas the present framework retains that structure throughout.

A key departure from the standard literature is the treatment of the Wirtinger operators themselves. Most presentations introduce $\partial f/\partial z$ and $\partial f/\partial \overline{z}$ as consequences of real-differentiability, derived by splitting the real partial derivatives into their combinations. Here the order is reversed: if $f$ admits a first-order approximation by a real-linear operator $L$ (Theorem 3), Lemma 2 forces $L$ to be widely-linear, and the Wirtinger operators arise as the unique coefficients of that approximation. The real partial derivatives are then derived consequences — $\partial f/\partial x_k = \partial f/\partial z_k + \partial f/\partial \overline{z}_k$ and

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$\partial f/\partial y_k = j(\partial f/\partial z_k - \partial f/\partial \overline{z}_k)$ — not prerequisites. Theorem 4 and Corollary 2 complete the circle: Lipschitz continuity of the Wirtinger operators is equivalent to real Fréchet differentiability with a Lipschitz derivative. The hypothesis is the same as in real calculus — Wirtinger differentiability in this sense is real Fréchet differentiability (Corollary 1) — but the operators are not a notational convenience obtained after realification: they are the coordinates in which first-order approximation of maps $\mathbb{C}^n \to \mathbb{C}^m$ is stated and computed natively.

Building on this foundation, the central results are a semilocal convergence theorem for NRW (Theorem 5), its affine-covariant version (Theorem 6) and a local quadratic convergence theorem (Theorem 7), all stated and proved in terms of $\partial f/\partial z$ and $\partial f/\partial \overline{z}$. The constants $\beta$, $\eta$, $L$ and the criterion $h = \beta L \eta \leq \frac{1}{2}$ are computable from the complex expression for $f$, with no expansion into real and imaginary parts. The affine-covariant constant $\omega$ is invariant under all invertible widely-linear transformations of the equations and never exceeds $\beta L$.

On the computational side, the NRW step has a single formula and two ways of computing it. The A-pivot and the B-pivot of the widely-linear inverse give the same step; the B-pivot is the A-pivot with $\overline{z}$ as the unknown, so $\Gamma$ and $\Sigma$ take two forms and the same algorithm serves both, the B-pivot iteration being carried out in $\overline{z}$ and conjugated only at the end. A- or B-dominance of the Wirtinger operators guarantees that the corresponding pivot applies, and at the two extremes, holomorphic and antiholomorphic maps, the step reduces to the pure Newton step, classical Newton-Raphson in $z$ or in $\overline{z}$. The same pure Newton step applied to the augmented system computes the general NRW step without pivoting, which is the natural choice when neither Wirtinger operator is cheap to invert. The cost comparison is not favorable in general: for dense operators NRW costs about four times as much as realified Newton-Raphson, and the Schur reduction is no cheaper than a factorization of the augmented system. The reduction pays off only when the pivot block, $\partial f/\partial z$ or $\partial f/\partial \overline{z}$, is cheap to invert, as when it is diagonal or sparse.

Two applications illustrated the results. In the single-generator infinite-bus power flow problem every constant was computed explicitly, the certificate held from both initial guesses ($h \approx 0.050$ and $h \approx 0.269$), and the iterates coincided with those of realified Newton-Raphson; the low-voltage root lies in the B-dominant region, where the B-pivot is the natural one. In vortex states of the discretized Gross-Pitaevskii equation, a phase pinning that changes one diagonal entry of each Wirtinger operator restored isolated roots, and Newton converged quadratically. The Kantorovich certificate, however, failed there by almost three orders of magnitude ($h \approx 293$), while the computational estimates of the affine-covariant constant were about three orders of magnitude smaller than $\beta L$: the Kantorovich condition is sufficient, not necessary.

The framework applies to any rootfinding problem over $\mathbb{C}^n$ where $f$ is real-differentiable but not holomorphic, a class that includes many nonlinear models in power systems, electromagnetics, signal processing and quantum mechanics. The main direction for future work is a rigorous, computable upper bound on $\omega$ expressed through the Wirtinger operators, which would turn the computational estimates of Section 7 into certificates. Exploiting problem-specific structure, such as that of multibus power flow, for sharper bounds and sparse implementations of the Schur reduction is another.

## Author contributions

All three authors contributed equally to all roles: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, and Writing – review & editing.

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## Data availability statement

No datasets were generated or analyzed during the current study.

## Code availability statement

The CBLAS/LAPACKE implementation, as well as the codes for the applications, can be found at https://github.com/gondoresearch/numerical_foundations_nrw_public.

## AI usage statement

Artificial intelligence tools were not used in the writing of this paper.

## Conflict of interest statement

The authors declare no conflicts of interest.

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