A Comprehensive Derivation of the Conditions of Conformality, Equivalence, and Equidistance of Normal Cylindrical Projections

Isaac Ramos, Leonard Silveira, Vinícius Martins, Robert Silva, Andréa Seixas, Sílvio Garnés, Lucas Calado, Paulo Airoldi

https://doi.org/10.1590/SciELOPreprints.16769

Submitted on: 2026-07-03
Posted on: 2026-07-07 (version 1)
(YYYY-MM-DD)

A Comprehensive Derivation of the Conditions of Conformality, Equivalence, and Equidistance of Normal Cylindrical Projections

Isaac Ramos Junior
Federal University of Pampa, Itaqui, Brazil
ORCID https://orcid.org/0009-0009-9657-4060

Leonard Niero da Silveira
Federal University of Pampa, Itaqui, Brazil
ORCID https://orcid.org/0000-0003-1780-135X

Vinícius Bergmann Martins
Federal University of Pampa, Itaqui, Brazil
ORCID https://orcid.org/0000-0001-8532-4247

Robert Martins da Silva
Federal University of Pampa, Itaqui, Brazil
ORCID https://orcid.org/0000-0002-0277-5980

Andréa de Seixas
Federal University of Pernambuco, Recife, Brazil
ORCID https://orcid.org/0000-0002-5879-4902

Sílvio Jacks dos Anjos Garnés
Federal University of Pernambuco, Recife, Brazil
ORCID https://orcid.org/0000-0002-0098-6645

Lucas Gonzales Lima Pereira Calado
Federal University of Pernambuco, Recife, Brazil
ORCID https://orcid.org/0000-0002-2456-8288

Paulo Alexandre de Borba Airoldi
Federal University of Pampa, Itaqui, Brazil
ORCID https://orcid.org/0009-0009-9648-9915

Abstract
Map projections play a significant role in representing the Earth's surface by converting coordinates defined on a curved reference surface into coordinates on a plane. Since no single map projection can preserve all geometric attributes at the same time, distortions related to distance, shape, and area inevitably arise, making the analysis of map projection properties essential in mathematical cartography. This study develops a detailed and rigorous derivation of the conditions for conformality, equivalence, and equidistance based on the first Gaussian fundamental quantities. As an illustrative case, the normal cylindrical projection tangent to the equator is considered, due to its historical relevance and its connection with well-known projections such as Mercator, Lambert cylindrical equal-area, and Plate Carrée. The derivations are formulated through a differential-geometric approach, linking surface metrics directly to cartographic distortions. The obtained results show that the classical properties of map projections can be systematically described using the first Gaussian fundamental quantities, thereby offering an integrated interpretation of their metric behavior. In this way, the paper emphasizes the importance of differential geometry for cartography and provides a broad mathematical and pedagogical discussion of the geometric principles that support map projections.

Keywords: Mathematical Cartography. Mapping. Distortion.

1 INTRODUCTION

Maps are a common feature of media, social networks, and public discussions, serving as powerful tools for communicating spatial information and supporting viewpoints. While map production was once largely limited to governments and political leaders, Web 2.0 technologies have made mapmaking widely accessible. Today, individuals, non-governmental organizations, and civic groups use maps to engage in public discourse and communicate their own perspectives and interests (Budke, 2025).

However, creating a map is far from trivial. On the contrary, the immense area of the Earth's surface and the physical and anthropogenic processes that occur on it must be compressed and represented on a small sheet of paper (Mocnik, 2025). Thus, it can be said that a map is produced by converting geospatial data from the Earth, which is in a spherical (or ellipsoidal) coordinate system, into a plane representation, whether on paper or in digital form. In this context, the process of transferring information from a curved surface to a plane surface is called map projection, and it can be conducted in several ways (Lapaine, 2025). From a more specific point of view, a map projection requires the transformation of a set of two independent coordinates in an Earth model, generally latitude φ and longitude λ, into a set of two independent coordinates on the map, such as the Cartesian coordinates x and y (Ghaderpour, 2016).

A notable characteristic of map projections, which is studied mathematically, is that all of them introduce distortions into the map, and their types are usually considered in terms of length, shape, and area (Pearson, 1990). This is relevant because the history of creating the best map projections can, in general, be regarded as the history of cartography itself, since every cartographer, when creating a map, first intuitively and later consciously, seeks to minimize its distortions (Novikova, 2025).

Map projections can be classified according to different criteria. One of them distinguishes the extrinsic aspect, which concerns the geometric relationship between the projection surface and the reference surface with regard to their nature (plane, conic, or cylindrical), type of contact (tangent or secant), and orientation (normal, transverse, or oblique), and the intrinsic aspect, which addresses the preserved properties as equidistance, conformality, or equal-area property (equivalence) and the method of construction (geometric, semi-geometric, or conventional). This classification provides a clear framework for understanding both the geometry and the purpose of projections (Richardus and Adler, 1972).

Nevertheless, regarding map projections, there is a classic difficulty that can be briefly illustrated. The Second World War raised new awareness of global problems and renewed interest in cartography, highlighting the need for greater attention to the subject. Nevertheless, even after 80 years, these issues remain relevant, although public understanding has advanced little. This difficulty is due to the intrinsic complexity of projections, linked to mathematics and to technical parameters that are not truly clear to laypeople, as well as to the way software and specialized literature commonly treat them as difficult to understand (Kessler, 2025).

Indeed, the mathematical basis of map projections is complex, which can be intimidating for some people. Although those familiar with advanced calculus find it easier to use detailed books on the subject, most users of geospatial data do not need to know the equations involved, since mapping software performs the calculations. Even so, having some understanding of the mathematics involved remains valuable (Kessler and Battersby, 2019). In this sense, among the many tools used in the study of the subject are the first Gaussian fundamental quantities. They are defined for specific surfaces of interest in map projections and are used both in deriving the projections themselves and in developing their distortions (Pearson, 1990).

Beyond that, recent contributions emphasize the educational value of student-centered cartography. Combining theoretical insights with practical classroom applications, they show how mapmaking can foster cartographic literacy, critical reflection, and participatory learning through approaches such as thematic maps, digital tools, mental maps, and map-based argumentation, while highlighting the role of teacher education in supporting these practices (Budke, 2025).

In this context, Morawski and Budke (2025) demonstrate how maps embedded in digital games can promote critical, reflective, and procedural cartographic literacy through a framework derived from the analysis of culturally influential games. Moser (2025) evaluates the use of choropleth maps in educational materials and digital environments, arguing that their pedagogical potential is often limited by insufficient methodological and didactic reflection. Edler et al. (2025) show that immersive virtual reality environments can enhance spatial thinking, critical decision-making, and engagement with real-world urban issues. Eberth, Wagener, and Zink (2025) illustrate how the integration of digital participation platforms and photovoice methods can strengthen map-production skills and critical map use in both teacher education and school settings. Research by Kunze et al. (2025) reveals that many geography students possess only a limited understanding of map-based argumentation prior to specific instruction, while Ulbricht and Lindau (2025) highlight the value of field trips and mental maps for shaping spatial conceptions and supporting teacher development. Similarly, Kunze and Budke (2025) find that geography teachers often have an incomplete understanding of map-based argumentation, underscoring the need for targeted training to better integrate these competencies into classroom practice.

For the reasons presented, this article seeks to provide a rigorous and detailed derivation of the equations associated with three fundamental properties of map projections, i.e., conformality, equal-area property, and equidistance, based on the first Gaussian fundamental quantities. Given the considerable number of existing map projections, the analysis focuses on the normal cylindrical projection tangent to the reference surface along the equator, owing to its historical significance and its connection to widely used projections such as Mercator, Lambert equal-area, and Plate Carrée. The methodology combines mathematical cartography with a structured and pedagogically oriented approach, emphasizing continuity in the derivations to minimize conceptual gaps and facilitate understanding. In this way, the study highlights not only the importance of mastering the mathematical foundations underlying map construction, but also the educational value of cartographic practice, demonstrating how mapmaking can serve as an accessible and effective teaching resource even in contexts that do not require advanced technological tools.

2 THE FIRST GAUSSIAN FUNDAMENTAL QUANTITIES

In general terms, classical differential geometry is the study of the local properties of curves and surfaces. Local properties are understood as those that depend only on the behavior of the curve or surface in the neighborhood of a point. The methods that have proved suitable for studying these properties are those of differential calculus (Carmo, 2016). This observation provides the foundation for the present work because, as will be demonstrated, the first Gaussian fundamental quantities are fundamentally grounded in differential geometry.

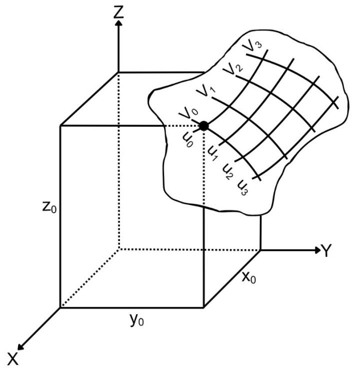
Thus, let there be an arbitrary surface associated with a three-dimensional Cartesian system, in which u and v are curves on that surface, as shown in Figure 1. In this way, the following parametric equations can be written for the Cartesian coordinates of any point on this surface, as shown in Equations (1), (2), and (3) (Richardus and Adler, 1972):

$$x = f_1(u, v) \tag{1}$$

$$y = f_2(u, v) \tag{2}$$

$$z = f_3(u, v) \tag{3}$$

Figure 1 – The *u* and *v* curves in an arbitrary surface.



Source: Adapted from Richardus and Adler (1972).

Taking the total differentials of Equations (1), (2), and (3) gives Equations (4), (5), and (6):

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \tag{4}$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \tag{5}$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \tag{6}$$

Now, let ($x + dx, y + dy, z + dz$) be the coordinates of another arbitrary point on this same surface, and let ds be the distance between it and the first point, considered to be infinitesimally small (Gauss, 1902). That is, ds represents an infinitesimally small element of a curve on any surface. Therefore, Equation (7) can be written:

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{7}$$

Substituting Equations (4), (5), and (6) into Equation (7) and expanding yields Equations (8), (9), and (10):

$$ds^2 = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)^2 + \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)^2 + \left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right)^2 \tag{8}$$

$$ds^2 = \left(\frac{\partial x}{\partial u}\right)^2 du^2 + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}dudv + \left(\frac{\partial x}{\partial v}\right)^2 dv^2 + \left(\frac{\partial y}{\partial u}\right)^2 du^2 + 2\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}dudv + \left(\frac{\partial y}{\partial v}\right)^2 dv^2 + \\ + \left(\frac{\partial z}{\partial u}\right)^2 du^2 + 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}dudv + \left(\frac{\partial z}{\partial v}\right)^2 dv^2 \quad (9)$$

$$ds^2 = \left(\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right) du^2 + 2\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right) dudv + \\ + \left(\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right) dv^2 \quad (10)$$

In Equation (10), the following considerations can be made, as shown in Equations (11), (12), and (13):

$$e = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \quad (11)$$

$$f = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \quad (12)$$

$$g = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \quad (13)$$

The quantities *e, f,* and *g* in Equations (11), (12), and (13) are known as the first Gaussian fundamental quantities. Substituting these coefficients back into Equation (10) yields Equation (14), which represents the differential line element of an arbitrary curve on a given surface (Pearson, 1990):

$$ds^2 = edu^2 + 2fdudv + gdv^2 \quad (14)$$

Something fundamental occurs with Equation (14), which is of great importance for the study of map projections, namely: when a surface is considered not as the boundary of a solid, but as a flexible body whose thickness is assumed to vanish, some properties of the surface depend on the form it assumes, whereas others are absolute and remain invariant regardless of the way in which the surface is bent. From this point of view, a plane surface and a surface that is developable onto a plane, such as cylindrical surfaces and conical surfaces, must be considered essentially identical, and the general method for comprehensively defining the nature of surfaces considered in this way is always based on this equation (Gauss, 1902). For these reasons, Equation (14) is the conceptual core of this work.

3 THE FUNDAMENTAL TRANSFORMATION MATRIX

Since the first Gaussian fundamental quantities for the reference surface (sphere or ellipsoid) are different from those belonging to the projection surface (plane, cone, or cylinder), in this text lowercase letters will be used for the first and uppercase letters for the second, as follows:

a) For the reference surface, according to Equations (15) to (21):

$$x = f_1(u, v) \quad (15)$$

$$y = f_2(u, v) \quad (16)$$

$$z = f_3(u, v) \quad (17)$$

$$ds^2 = edu^2 + 2fdudv + gdv^2 \quad (18)$$

$$e = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \quad (19)$$

$$f = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \quad (20)$$

$$g = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \tag{21}$$

b) For the projection surface, according to Equation (22) to (28):

$$X = g_1(U, V) \tag{22}$$

$$Y = g_2(U, V) \tag{23}$$

$$Z = g_3(U, V) \tag{24}$$

$$dS^2 = EdU^2 + 2FdUdV + GdV^2 \tag{25}$$

$$E = \left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 + \left(\frac{\partial Z}{\partial U}\right)^2 \tag{26}$$

$$F = \frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V} \tag{27}$$

$$G = \left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 + \left(\frac{\partial Z}{\partial V}\right)^2 \tag{28}$$

Equations (15) to (28) show that, up to this point, only Gaussian quantities have been found separately for two types of surfaces, namely the reference surface and the projection surface; that is, *e, f, g* for the reference surface and *E, F, G* for the projection surface. A procedure for relating them will now be introduced.

Bearing in mind that the purpose of a map projection is to represent a curved surface on a plane surface, that is, to represent a reference surface on a projection surface, this means that, mathematically, one seeks to find both *U* and *V* as functions of *u* and *v*, as shown in Equations (29) and (30):

$$U = h_1(u, v) \tag{29}$$

$$V = h_2(u, v) \tag{30}$$

The reverse relationship must also occur, that is, since map projections are reversible (Richardus and Adler, 1972), Equations (31) and (32) should also be valid:

$$u = h_3(U, V) \tag{31}$$

$$v = h_4(U, V) \tag{32}$$

Consequently, one can replace Equations (29) and (30) into Equations (22), (23), and (24), obtaining Equations (33), (34), and (35):

$$X = g_1(U, V) = g_1\big(h_1(u, v), h_2(u, v)\big) = g_4(u, v) \tag{33}$$

$$Y = g_2(U, V) = g_2\big(h_1(u, v), h_2(u, v)\big) = g_5(u, v) \tag{34}$$

$$Z = g_3(U, V) = g_3\big(h_1(u, v), h_2(u, v)\big) = g_6(u, v) \tag{35}$$

Now, the first Gaussian fundamental quantities can be written for the functions that relates the two types of surfaces, which will be different from those already found. That is why $\bar{E}, \bar{F}, \bar{G}$ will be used to represent them, forming a new set of equations, as shown in Equation (36) to (42):

$$X = g_4(u, v) \tag{36}$$

$$Y = g_5(u, v) \tag{37}$$

$$Z = g_6(u, v) \tag{38}$$

$$dS^2 = \bar{E}du^2 + 2\bar{F}dudv + \bar{G}dv^2 \tag{39}$$

$$\bar{E} = \left(\frac{\partial X}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial u}\right)^2 + \left(\frac{\partial Z}{\partial u}\right)^2 \tag{40}$$

$$\bar{F} = \frac{\partial X}{\partial u}\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial u}\frac{\partial Y}{\partial v} + \frac{\partial Z}{\partial u}\frac{\partial Z}{\partial v} \tag{41}$$

$$\bar{G} = \left(\frac{\partial X}{\partial v}\right)^2 + \left(\frac{\partial Y}{\partial v}\right)^2 + \left(\frac{\partial Z}{\partial v}\right)^2 \tag{42}$$

It is worth noting that the element dS in Equation (25) and in Equation (39) are the same, that is a differential element of a curve on the projection surface. However, in Equation (39) it is represented by X, Y, Z as functions of u, v, while in Equation (25) it is represented by X, Y, Z as functions of U, V.

Now it is possible to find the partial derivatives related to the equations of $\bar{E}, \bar{F}, \bar{G}$. This can be done by applying the chain rule for composite functions in Equations (33), (34), and (35), according to Equations (43) to (48):

$$\frac{\partial X}{\partial u} = \frac{\partial X}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial u} \tag{43}$$

$$\frac{\partial X}{\partial v} = \frac{\partial X}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial v} \tag{44}$$

$$\frac{\partial Y}{\partial u} = \frac{\partial Y}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial Y}{\partial V}\frac{\partial V}{\partial u} \tag{45}$$

$$\frac{\partial Y}{\partial v} = \frac{\partial Y}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial Y}{\partial V}\frac{\partial V}{\partial v} \tag{46}$$

$$\frac{\partial Z}{\partial u} = \frac{\partial Z}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial Z}{\partial V}\frac{\partial V}{\partial u} \tag{47}$$

$$\frac{\partial Z}{\partial v} = \frac{\partial Z}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial Z}{\partial V}\frac{\partial V}{\partial v} \tag{48}$$

Equations (43) to (48) can be replaced into Equations (40), (41), and (42), and can be expanded according to Equations (49), (50), and (51):

$$\begin{aligned}
\bar{E} &= \left(\frac{\partial X}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial Y}{\partial V}\frac{\partial V}{\partial u}\right)^2 + \left(\frac{\partial Z}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial Z}{\partial V}\frac{\partial V}{\partial u}\right)^2 = \\
&= \left(\frac{\partial X}{\partial U}\frac{\partial U}{\partial u}\right)^2 + 2\left(\frac{\partial X}{\partial U}\frac{\partial X}{\partial V}\frac{\partial U}{\partial u}\frac{\partial V}{\partial u}\right) + \left(\frac{\partial X}{\partial V}\frac{\partial V}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial U}\frac{\partial U}{\partial u}\right)^2 + 2\left(\frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V}\frac{\partial U}{\partial u}\frac{\partial V}{\partial u}\right) + \left(\frac{\partial Y}{\partial V}\frac{\partial V}{\partial u}\right)^2 + \\
&\quad + \left(\frac{\partial Z}{\partial U}\frac{\partial U}{\partial u}\right)^2 + 2\left(\frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\frac{\partial U}{\partial u}\frac{\partial V}{\partial u}\right) + \left(\frac{\partial Z}{\partial V}\frac{\partial V}{\partial u}\right)^2 = \\
&= \left(\left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 + \left(\frac{\partial Z}{\partial U}\right)^2\right)\left(\frac{\partial U}{\partial u}\right)^2 + 2\left(\frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\right)\frac{\partial U}{\partial u}\frac{\partial V}{\partial u} + \\
&\quad + \left(\left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 + \left(\frac{\partial Z}{\partial V}\right)^2\right)\left(\frac{\partial V}{\partial u}\right)^2 = \\
&= E\left(\frac{\partial U}{\partial u}\right)^2 + 2F\frac{\partial U}{\partial u}\frac{\partial V}{\partial u} + G\left(\frac{\partial V}{\partial u}\right)^2
\end{aligned} \tag{49}$$

$$\begin{aligned}
\bar{F} &= \left(\frac{\partial X}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial u}\right)\left(\frac{\partial X}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial v}\right) + \left(\frac{\partial Y}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial Y}{\partial V}\frac{\partial V}{\partial u}\right)\left(\frac{\partial Y}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial Y}{\partial V}\frac{\partial V}{\partial v}\right) + \\
&\quad + \left(\frac{\partial Z}{\partial U}\frac{\partial U}{\partial u} + \frac{\partial Z}{\partial V}\frac{\partial V}{\partial u}\right)\left(\frac{\partial Z}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial Z}{\partial V}\frac{\partial V}{\partial v}\right) =
\end{aligned} \tag{50}$$

$$
\begin{aligned}
&= \left(\frac{\partial X}{\partial U}\right)^2 \frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + \frac{\partial X}{\partial U}\frac{\partial X}{\partial V}\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} + \frac{\partial X}{\partial U}\frac{\partial X}{\partial V}\frac{\partial V}{\partial u}\frac{\partial U}{\partial v} + \left(\frac{\partial X}{\partial V}\right)^2 \frac{\partial V}{\partial u}\frac{\partial V}{\partial v} + \left(\frac{\partial Y}{\partial U}\right)^2 \frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + \\
&+ \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V}\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V}\frac{\partial V}{\partial u}\frac{\partial U}{\partial v} + \left(\frac{\partial Y}{\partial V}\right)^2 \frac{\partial V}{\partial u}\frac{\partial V}{\partial v} + \left(\frac{\partial Z}{\partial U}\right)^2 \frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + \\
&+ \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\frac{\partial V}{\partial u}\frac{\partial U}{\partial v} + \left(\frac{\partial Z}{\partial V}\right)^2 \frac{\partial V}{\partial u}\frac{\partial V}{\partial v} = \\
&= \left(\left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 + \left(\frac{\partial Z}{\partial U}\right)^2\right)\frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + \left(\frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\right)\left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v}\right) + \\
&+ \left(\frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\right)\left(\frac{\partial V}{\partial u}\frac{\partial U}{\partial v}\right) + \left(\left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 + \left(\frac{\partial Z}{\partial V}\right)^2\right)\left(\frac{\partial V}{\partial u}\frac{\partial V}{\partial v}\right) = \\
&= \left(\left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 + \left(\frac{\partial Z}{\partial U}\right)^2\right)\frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + \left(\frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\right)\left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} + \frac{\partial V}{\partial u}\frac{\partial U}{\partial v}\right) + \\
&+ \left(\left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 + \left(\frac{\partial Z}{\partial V}\right)^2\right)\left(\frac{\partial V}{\partial u}\frac{\partial V}{\partial v}\right) = \\
&= E\frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + F\left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} + \frac{\partial V}{\partial u}\frac{\partial U}{\partial v}\right) + G\left(\frac{\partial V}{\partial u}\frac{\partial V}{\partial v}\right) \\
&\bar{G} = \left(\frac{\partial X}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial X}{\partial V}\frac{\partial V}{\partial v}\right)^2 + \left(\frac{\partial Y}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial Y}{\partial V}\frac{\partial V}{\partial v}\right)^2 + \left(\frac{\partial Z}{\partial U}\frac{\partial U}{\partial v} + \frac{\partial Z}{\partial V}\frac{\partial V}{\partial v}\right)^2 = \\
&= \left(\frac{\partial X}{\partial U}\frac{\partial U}{\partial v}\right)^2 + 2\left(\frac{\partial X}{\partial U}\frac{\partial X}{\partial V}\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}\right) + \left(\frac{\partial X}{\partial V}\frac{\partial V}{\partial v}\right)^2 + \left(\frac{\partial Y}{\partial U}\frac{\partial U}{\partial v}\right)^2 + 2\left(\frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V}\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}\right) + \left(\frac{\partial Y}{\partial V}\frac{\partial V}{\partial v}\right)^2 + \\
&+ \left(\frac{\partial Z}{\partial U}\frac{\partial U}{\partial v}\right)^2 + 2\left(\frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}\right) + \left(\frac{\partial Z}{\partial V}\frac{\partial V}{\partial v}\right)^2 = \\
&= \left(\left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 + \left(\frac{\partial Z}{\partial U}\right)^2\right)\left(\frac{\partial U}{\partial v}\right)^2 + 2\left(\frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V}\right)\left(\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}\right) + \\
&+ \left(\left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 + \left(\frac{\partial Z}{\partial V}\right)^2\right)\left(\frac{\partial V}{\partial v}\right)^2 = \\
&= E\left(\frac{\partial U}{\partial v}\right)^2 + 2F\left(\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}\right) + G\left(\frac{\partial V}{\partial v}\right)^2
\end{aligned}
\tag{51}
$$

Now, one can write the results according to Equations (52), (53), and (54):

$$\bar{E} = E\left(\frac{\partial U}{\partial u}\right)^2 + 2F\frac{\partial U}{\partial u}\frac{\partial V}{\partial u} + G\left(\frac{\partial V}{\partial u}\right)^2 \tag{52}$$

$$\bar{F} = E\frac{\partial U}{\partial u}\frac{\partial U}{\partial v} + F\left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} + \frac{\partial V}{\partial u}\frac{\partial U}{\partial v}\right) + G\left(\frac{\partial V}{\partial u}\frac{\partial V}{\partial v}\right) \tag{53}$$

$$\bar{G} = E\left(\frac{\partial U}{\partial v}\right)^2 + 2F\left(\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}\right) + G\left(\frac{\partial V}{\partial v}\right)^2 \tag{54}$$

Equations (52), (53), and (54) are normally expressed in matrix form, according to Equation (55) (Pearson, 1990):

$$\begin{bmatrix}\bar{E}\\ \bar{F}\\ \bar{G}\end{bmatrix} = \begin{bmatrix}\left(\frac{\partial U}{\partial u}\right)^2 & 2\frac{\partial U}{\partial u}\frac{\partial V}{\partial u} & \left(\frac{\partial V}{\partial u}\right)^2\\ \frac{\partial U}{\partial u}\frac{\partial U}{\partial v} & \frac{\partial U}{\partial u}\frac{\partial V}{\partial v}+\frac{\partial V}{\partial u}\frac{\partial U}{\partial v} & \frac{\partial V}{\partial u}\frac{\partial V}{\partial v}\\ \left(\frac{\partial U}{\partial v}\right)^2 & 2\left(\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}\right) & \left(\frac{\partial V}{\partial v}\right)^2\end{bmatrix}\begin{bmatrix}E\\ F\\ G\end{bmatrix} \tag{55}$$

The 3x3 matrix in Equation (55) is known as the fundamental transformation matrix (Richardus and Adler, 1972), and it is highly relevant in the development of map projections.

3.1 Application of the fundamental transformation matrix on the reference and projection surfaces

With the fundamental transformation matrix available, it is possible to determine the first Gaussian fundamental quantities for the reference and projection surfaces. When the sphere is used as the reference surface, its parametric equations are initially employed, as shown in Equations (56), (57), and (58) (Lapaine, 2025):

$$x = Rcos\varphi cos\lambda \tag{56}$$

$$y = Rcos\varphi sin\lambda \tag{57}$$

$$z = Rsin\varphi \tag{58}$$

In Equations (56), (57) e (58):

x, y, z are the Cartesian coordinates of any point on the surface of the sphere;
φ is the latitude;
λ is the longitude;
R is the radius of the sphere.

In the case of the sphere, we have that $u = \varphi$ and $v = \lambda$ (Ramos Junior et al., 2025). So, the following substitutions can be made in Equations (15) to (21), arriving at Equations (59) to (65):

$$x = f_1(\varphi, \lambda) \tag{59}$$

$$y = f_2(\varphi, \lambda) \tag{60}$$

$$z = f_3(\varphi, \lambda) \tag{61}$$

$$ds^2 = ed\varphi^2 + 2fd\varphi d\lambda + gd\lambda^2 \tag{62}$$

$$e = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 \tag{63}$$

$$f = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda} + \frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \lambda} \tag{64}$$

$$g = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 \tag{65}$$

By deriving Equations (56), (57), and (58) and substituting according to Equations (63), (64), and (65), we get Equations (66), (67), and (68), according to (Pearson, 1990):

$$e = R^2 \tag{66}$$

$$f = 0 \tag{67}$$

$$g = R^2cos^2\varphi \tag{68}$$

Following the same reasoning, in the case where the ellipsoid is used as the reference surface, the results are the constants in Equations (69) to (73) (Richardus and Adler, 1972):

$$e = M^2 \tag{69}$$

$$f = 0 \tag{70}$$

$$g = N^2 cos^2\varphi \tag{71}$$

$$M = \frac{a(1-\epsilon^2)}{\sqrt{(1-\epsilon^2 sin^2\varphi)^3}} \tag{72}$$

$$N = \frac{a}{\sqrt{1-\epsilon^2 sin^2\varphi}} \tag{73}$$

In Equations (69) a (73):
M is the radius of curvature in the meridian section passing through any point on the surface of the ellipsoid;
N is the length of the normal to the ellipsoid from the point on the surface of the ellipsoid to its minor axis;
a is the major semiaxes of the ellipsoid;
ϵ is the first eccentricity.

It's worth noting that, in this text, the Greek letter ϵ (epsilon) was used for the first eccentricity because the letter e from the English alphabet had already been used earlier to represent one of the first Gaussian fundamental quantities. However, this naming is not exclusive, since, according to Meyer (2018, p. 65), "ϵ is the symbol for eccentricity both in common literature and as adopted by the International Organization for Standardization (ISO) and the International Electrotechnical Commission (IEC), although other authors use different symbols, like e".

Thus, the same procedure can be followed to find the first Gaussian fundamental quantities with respect to the projection surface, which in this work takes as its object the normal cylinder tangent to the reference surface. A cylinder in normal position means that, according to Santos (1985, p. 121), "the axis of symmetry of the cylinder coincides with the Earth's axis of rotation".

From this perspective, in normal cylindrical projections, lines of longitude depend only on longitude, and lines of latitude depend only on latitude. It is convenient to require that the origin of the system (U, V) coincides with the corresponding origin of the parametric curves of the reference surface. Thus, when coordinates of the type (φ, λ) are used in the parametrization, the coordinates (U, V) must be interpreted as rectangular coordinates (X, Y) in the plane (Tobler, 1962). In these projections, meridians are represented by straight, parallel, and equally spaced lines, and parallels by straight, parallel lines orthogonal to the meridians. They may be conformal, equal-area, arbitrary, or, in particular cases, equidistant along the meridians (Bugayevskiy and Snyder, 1995). Therefore, the parametric equations for the normal cylinder can be written as shown in Equations (74), (75), and (76):

$$X = g_1(U, V) = U \tag{74}$$

$$Y = g_2(U, V) = V \tag{75}$$

$$Z = g_3(U, V) = 0 \tag{76}$$

By substituting Equations (74), (75), and (76) into Equations (26), (27), and (28), we get the first Gaussian fundamental quantities for the normal cylinder, according to Equations (77), (78), and (79):

$$E = \left(\frac{\partial X}{\partial U}\right)^2 + \left(\frac{\partial Y}{\partial U}\right)^2 + \left(\frac{\partial Z}{\partial U}\right)^2 = 1 + 0 + 0 = 1 \tag{77}$$

$$F = \frac{\partial X}{\partial U}\frac{\partial X}{\partial V} + \frac{\partial Y}{\partial U}\frac{\partial Y}{\partial V} + \frac{\partial Z}{\partial U}\frac{\partial Z}{\partial V} = 0 + 0 + 0 = 0 \tag{78}$$

$$G = \left(\frac{\partial X}{\partial V}\right)^2 + \left(\frac{\partial Y}{\partial V}\right)^2 + \left(\frac{\partial Z}{\partial V}\right)^2 = 0 + 1 + 0 = 1 \tag{79}$$

With the first fundamental Gaussian quantities given in Equations (63), (64), and (65), and in Equations (77), (78), and (79), one can apply it to rewrite the fundamental transformation matrix given in Equation (55), as shown in Equation (80):

$$\begin{bmatrix} \bar{E} \\ \bar{F} \\ \bar{G} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial X}{\partial \varphi}\right)^2 & 2\frac{\partial X}{\partial \varphi}\frac{\partial Y}{\partial \varphi} & \left(\frac{\partial Y}{\partial \varphi}\right)^2 \\ \frac{\partial X}{\partial \varphi}\frac{\partial X}{\partial \lambda} & \frac{\partial X}{\partial \varphi}\frac{\partial Y}{\partial \lambda} + \frac{\partial Y}{\partial \varphi}\frac{\partial X}{\partial \lambda} & \frac{\partial Y}{\partial \varphi}\frac{\partial Y}{\partial \lambda} \\ \left(\frac{\partial X}{\partial \lambda}\right)^2 & 2\left(\frac{\partial X}{\partial \lambda}\frac{\partial Y}{\partial \lambda}\right) & \left(\frac{\partial Y}{\partial \lambda}\right)^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{80}$$

Therefore, in normal cylindrical projections, since the lines of the same longitude λ depend only on the longitude, so that $X = j_1(\lambda)$, and since the lines of the same latitude φ depend only on the latitude, so that $Y = j_2(\varphi)$, we can write Equations (81) and (82) in the following way:

$$X = j_1(u, v) = j_1(v) = j_1(\lambda) \tag{81}$$

$$Y = j_2(u, v) = j_2(u) = j_2(\varphi) \tag{82}$$

Among the partial derivatives of Equations (81) and (82), notice that two of them are equal to zero, as shown by Equations (83) and (84):

$$\frac{\partial X}{\partial \varphi} = 0 \tag{83}$$

$$\frac{\partial Y}{\partial \lambda} = 0 \tag{84}$$

With that, by substituting Equations (83) and (84) into Equation (80) and breaking it down, we get Equations (85), (86), (87), and (88), which depend on variables from both the reference surface and the projection surface:

$$\begin{bmatrix} \bar{E} \\ \bar{F} \\ \bar{G} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \left(\frac{\partial Y}{\partial \varphi}\right)^2 \\ 0 & \frac{\partial Y}{\partial \varphi}\frac{\partial X}{\partial \lambda} & 0 \\ \left(\frac{\partial X}{\partial \lambda}\right)^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{85}$$

$$\bar{E} = \left(\frac{\partial Y}{\partial \varphi}\right)^2 \tag{86}$$

$$\bar{F} = 0 \tag{87}$$

$$\bar{G} = \left(\frac{\partial X}{\partial \lambda}\right)^2 \tag{88}$$

4 THE LINEAR SCALE FACTOR

The linear scale factor, or simply scale factor, refers to the relative length of a line in the Earth model and the same line represented on the map (Pearson, 1990). Or, according to Krakiwsky (1973, p. 30), “The scale factor describes, at each point of the map projection, the amount of length distortion”.

Mathematically, the linear scale factor is the ratio between the differential element of the curve on the plane of the projection surface and the differential element of the corresponding curve on the reference surface (Lapaine, 2021). Thus, to find a general equation, Equation (89) can be written:

$$m = \frac{dS}{ds} \tag{89}$$

To express the scale factor in terms of first Gaussian fundamental quantities, Equation (89) can be squared, and then Equations (14) and (39) can be substituted into the resulting equation, yielding Equation (90):

$$m^2 = \frac{dS^2}{ds^2} = \frac{\bar{E}du^2 + 2\bar{F}dudv + \bar{G}dv^2}{edu^2 + 2fdudv + gdv^2} \tag{90}$$

From this moment, Equation (90) can be developed in many ways. In one of them, we can divide the numerator and the denominator by dv^2, and develop it, as shown in Equations (91) and (92):

$$m^2 = \frac{\bar{E}\frac{du^2}{dv^2} + \frac{2\bar{F}dudv}{dv^2} + \frac{\bar{G}dv^2}{dv^2}}{e\frac{du^2}{dv^2} + \frac{2fdudv}{dv^2} + \frac{gdv^2}{dv^2}} \tag{91}$$

$$m^2 = \frac{\bar{E}\left(\frac{du}{dv}\right)^2 + 2\bar{F}\frac{du}{dv} + \bar{G}}{e\left(\frac{du}{dv}\right)^2 + 2f\frac{du}{dv} + g} \tag{92}$$

The reason for the division by dv^2 is to make it clear that *m* depends on the direction of $\frac{du}{dv}$. Note that in Equation (92), considering the sphere (or the ellipsoid) and the cylinder, one can make the substitution of $\bar{F} = f = 0$, and as we have already seen that $u = \varphi$ and $v = \lambda$, we arrive at Equation (93):

$$m^2 = \frac{\bar{E}\left(\frac{d\varphi}{d\lambda}\right)^2 + \bar{G}}{e\left(\frac{d\varphi}{d\lambda}\right)^2 + g} \tag{93}$$

Regarding the differential elements $d\varphi$ and $d\lambda$ given in Equation (93), where the first refers to the parallel and the second to the meridian, it is worth noting that, in our analysis, $d\varphi$ depends only on latitude, and $d\lambda$ depends only on longitude. In this way, we can say that a differential change in latitude alone will generate a differential element of a curve that depends only on φ, which will be called ds_φ, and, by similar reasoning, the differential element of a curve generated solely by a differential change in longitude will be called ds_λ. This geometry can be seen through the differential quadrilateral in Figure 2.

5 THE DIFFERENTIAL PARALLELOGRAM

The visualization of a differential parallelogram is straightforward and can be carried out by following Figure 2 as follows. Imagine two points *P* and *Q* on an arbitrary surface, infinitesimally close to each other. From these points, coordinate lines are drawn, forming a parallelogram with infinitesimally small dimensions, used to evaluate local deformations related to distances, area, and angles that occur when the Earth's surface is projected onto the plane.

With this, Equations (94) and (95) can be written:

$$P = (\varphi, \lambda) \tag{94}$$

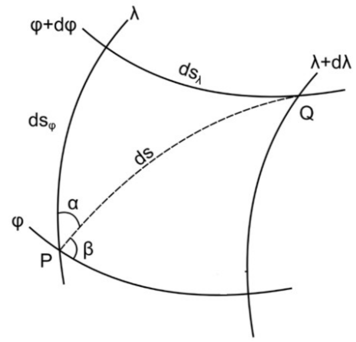
$$Q = (\varphi + d\varphi, \lambda + d\lambda) \tag{95}$$

This means that, from Figure 2, one can write Equations (96) and (97):

$$ds_\varphi^2 = ed\varphi^2 + 2fd\varphi d\lambda + gd\lambda^2 \tag{96}$$

$$ds_\lambda^2 = ed\varphi^2 + 2fd\varphi d\lambda + gd\lambda^2 \tag{97}$$

Figure 2 – Differential parallelogram.



Source: Adapted from Ramos Junior et al. (2025)

Since it has already been deduced that, for the sphere (or ellipsoid), $f = 0$, then from Equations (96) and (97) one can write Equations (98) and (99):

$$ds_\varphi^2 = ed\varphi^2 + gd\lambda^2 \tag{98}$$

$$ds_\lambda^2 = ed\varphi^2 + gd\lambda^2 \tag{99}$$

And now, since ds_φ depends only on φ, it means that in this case $d\lambda = 0$, and since ds_λ depends only on λ, it means that in this case $d\varphi = 0$. In this way, from Equations (98) and (99) one can write Equations (100) and (101) and simplify them according to Equations (102) and (103):

$$ds_\varphi^2 = ed\varphi^2 \tag{100}$$

$$ds_\lambda^2 = gd\lambda^2 \tag{101}$$

$$ds_\varphi = \sqrt{e}d\varphi \tag{102}$$

$$ds_\lambda = \sqrt{g}d\lambda \tag{103}$$

With the help of Figure 2 and Equations (102) and (103), you can notice that there is a differential triangle whose sides are ds, $\sqrt{e}d\varphi$, and $\sqrt{g}d\lambda$. This means that you can apply the Law of Cosines. To do this, let α be the angle opposite the side $\sqrt{g}d\lambda$, and β the angle opposite the side $\sqrt{e}d\varphi$, and let 180 - (α+β) be the angle opposite the side ds. Then we can write Equations (104), (105), and (106):

$$ds^2 = \left(\sqrt{e}d\varphi\right)^2 + \left(\sqrt{g}d\lambda\right)^2 - 2\sqrt{eg}d\varphi d\lambda\cos(180 - (\alpha + \beta)) \tag{104}$$

$$ds^2 = ed\varphi^2 + gd\lambda^2 - 2\sqrt{eg}d\varphi d\lambda\cos(180 - (\alpha + \beta)) \tag{105}$$

$$ds^2 = ed\varphi^2 + gd\lambda^2 + 2\sqrt{eg}d\varphi d\lambda cos(\alpha + \beta) \tag{106}$$

By equating Equation (106) with Equation (14), and working it out, we arrive at Equations (107), (108), and (109):

$$ed\varphi^2 + gd\lambda^2 + 2\sqrt{eg}d\varphi d\lambda cos(\alpha + \beta) = ed\varphi^2 + 2fd\varphi d\lambda + gd\lambda^2 \tag{107}$$

$$2\sqrt{eg}d\varphi d\lambda cos(\alpha + \beta) = 2fd\varphi d\lambda \tag{108}$$

$$cos(\alpha + \beta) = \frac{f}{\sqrt{eg}} \tag{109}$$

In classical texts, like in Gauss (1902, p. 25), in Equation (109) the term (α + β) is usually called ω. Doing it this way, we get to Equation (110):

$$cos\omega = \frac{f}{\sqrt{eg}} \tag{110}$$

Note in Equation (110) that, since for the sphere (or ellipsoid) $f = 0$, this means that the angle between any meridian and any parallel is a right angle. In other words, parallels and meridians intersect at right angles. This also occurs in the case of the cylinder, since it has $F = 0$. Thus, analogously, Equation (111) can be written for the projection surface, using uppercase letters:

$$cos\Omega = \frac{F}{\sqrt{EG}} \tag{111}$$

As it has been shown that the curves intersect at a right angle for the sphere, the ellipsoid, and the cylinder, one can work with differential right triangles. In other words, we can now say that we can manipulate a differential right triangle whose hypotenuse is ds and whose legs are $\sqrt{e}d\varphi$ and $\sqrt{g}d\lambda$. Thus, we can write Equations (112) and (113):

$$tg\alpha = \frac{\sqrt{g}d\lambda}{\sqrt{e}d\varphi} \tag{112}$$

$$\frac{d\varphi}{d\lambda} = \frac{1}{tg\alpha}\sqrt{\frac{g}{e}} \tag{113}$$

6 CONDITIONS OF CONFORMALITY

Projections in which the shape of small elements are preserved are called conformal (Melluish, 2014). This means that working with the first Gaussian fundamental quantities in this case is perfectly feasible. So, to start the procedure for deriving the conditions of conformality of cylindrical projections, we can initially substitute Equation (113) into Equation (93), obtaining Equations (114) and (115):

$$m^2 = \frac{\bar{E}\left(\frac{d\varphi}{d\lambda}\right)^2 + \bar{G}}{e\left(\frac{d\varphi}{d\lambda}\right)^2 + g} \tag{114}$$

$$m^2 = \frac{\bar{E}\left(\frac{1}{tg\alpha}\sqrt{\frac{g}{e}}\right)^2 + \bar{G}}{e\left(\frac{1}{tg\alpha}\sqrt{\frac{g}{e}}\right)^2 + g} \tag{115}$$

Dividing the numerator and the denominator of Equation (115) by g and developing it leads to Equations (116) and (117):

$$m^2 = \frac{\frac{\bar{E}}{e}\frac{1}{tg^2\alpha} + \frac{\bar{G}}{g}}{\frac{1}{tg^2\alpha} + 1} \tag{116}$$

$$m^2 = \frac{\frac{\bar{E}}{e}\frac{cos^2\alpha}{sin^2\alpha} + \frac{\bar{G}}{g}}{\frac{cos^2\alpha}{sin^2\alpha} + 1} \tag{117}$$

Multiplying the numerator and denominator of Equation (117) by $sin^2\alpha$ and expanding, we get Equations (118) and (119):

$$m^2 = \frac{\frac{\bar{E}}{e}cos^2\alpha + \frac{\bar{G}}{g}sin^2\alpha}{cos^2\alpha + sin^2\alpha} \tag{118}$$

$$m^2 = \frac{\bar{E}}{e}cos^2\alpha + \frac{\bar{G}}{g}sin^2\alpha \tag{119}$$

Looking at Equation (119), one can see that it depends on the angle α, which is the azimuth from *P* to *Q*, as shown in Figure 2. To prevent this from happening, that is, for Equation (119) to no longer depend on any direction, consider Equation (120):

$$\frac{\bar{E}}{e} = \frac{\bar{G}}{g} = K \tag{120}$$

It follows that, if Equation (120) is substituted into Equation (119), *m* will assume the same value for any direction with respect to that point on the projection surface, or, according to Canters and Decleir (1989, p. 07), “Conformal projections are obtained when the scale factor is independent of azimuth or is the same in all directions”. Algebraically, this substitution can be made as shown in Equations (121), (122), and (123):

$$m^2 = Kcos^2\alpha + Ksin^2\alpha \tag{121}$$

$$m^2 = K(cos^2\alpha + sin^2\alpha) \tag{122}$$

$$m^2 = K \tag{123}$$

Accordingly, using the first Gaussian fundamental quantities, it can be shown that a normal cylindrical projection is conformal if it satisfies the conditions given by Equations (124) and (125):

$$\bar{F} = f = 0 \tag{124}$$

$$\frac{\bar{E}}{e} = \frac{\bar{G}}{g} = K \tag{125}$$

Intrinsically to Equation (124), one further observation can be made. Since it was used in demonstrating the perpendicularity between meridians and parallels, it is worth emphasizing a peculiarity, namely: if this right-angle intersection on the reference surface is not maintained on the projection surface, under no circumstances can shape be preserved (Roblin, 1969). Thus, the condition in Equation (124) is necessary, but not sufficient for conformality, since the condition in Equation (125) must also be satisfied.

7 CONDITION OF EQUIVALENCE

In Figure 2, the area of the parallelogram will be bounded by the two lines of the first system corresponding to φ and φ + *d*φ, and by the two lines of the second system corresponding to λ and λ + *d*λ (Gauss, 1902), and according to Canters and Decleir (1989, p. 08) “if, during the mapping process, the elementary areas are preserved, the projection is called equivalent”.

Since Figure 2 deals with differential elements, their dimensions allow the use of Euclidean geometry equations. So, to find the condition of equal-area property, let the area of the differential element in Figure 2 be represented by Equations (126) and (127) at first:

$$da = \sqrt{e}d\varphi\sqrt{g}d\lambda sin\omega \tag{126}$$

$$da = \sqrt{eg}sin\omega d\varphi d\lambda \tag{127}$$

Considering that Equation (110) represents *cos*ω, some manipulation can be done, according to Equations (128), (129), and (130):

$$cos\omega = \sqrt{1 - sin^2\omega} = \frac{f}{\sqrt{eg}} \tag{128}$$

$$sin\omega = \sqrt{1 - \frac{f^2}{eg}} \tag{129}$$

$$sin\omega = \sqrt{\frac{eg - f^2}{eg}} \tag{130}$$

Equation (130) has a term that often appears in mathematical cartography, which will be highlighted according to Equation (131) (Bugayevskiy and Snyder, 1995):

$$h^2 = eg - f^2 \tag{131}$$

Hence, by substituting Equation (131) into Equation (130), Equation (132) can be written:

$$sin\omega = \frac{h}{\sqrt{eg}} \tag{132}$$

With the help of Equation (132), Equation (127) can be rewritten as Equation (133):

$$da = \sqrt{eg} sin\omega d\varphi d\lambda = hd\varphi d\lambda \tag{133}$$

Equation (133) can also be written according to Gauss's original notation (Gauss, 1902), by substituting Equation (131) into Equation (133), as shown in Equation (134):

$$da = \sqrt{eg - f^2} d\varphi d\lambda \tag{134}$$

Similarly to Equations (131) and (133), one can write Equations (135) and (136), regarding the projection surface:

$$\bar{H}^{\,2} = \bar{E}\bar{G} - \bar{F}^{\,2} \tag{135}$$

$$dA = \bar{H} d\varphi d\lambda = \sqrt{EG - F^2} d\varphi d\lambda \tag{136}$$

The scale factor for area at a given point is the ratio between the area of an infinitesimal quadrilateral on the projection and the area of the corresponding infinitesimal quadrilateral on the reference surface (Bugayevskiy and Snyder, 1995). Thus, with the differential areas of the different surfaces given in Equations (133) and (136), the scale factor for area, here denoted by m_a, can be obtained, as shown in Equation (137):

$$m_a = \frac{dA}{da} \tag{137}$$

In Equation (137), if the area on the map is equal to the area on the sphere (or on the ellipsoid), the projection is equivalent (Pearson, 1990). In other words, equal-area property occurs when the scale factor for the area is equal to 1. Algebraically, Equation (138) can be written:

$$\frac{dA}{da} = 1 \tag{138}$$

By substituting Equations (133) and (136) into Equation (138), we get Equations (139) and (140):

$$\frac{\bar{H} d\varphi d\lambda}{h d\varphi d\lambda} = 1 \tag{139}$$

$$\frac{\bar{H}}{h} = 1 \tag{140}$$

Equation (140) is a representation of the condition of equal-area property of normal cylindrical projections using the first Gaussian fundamental quantities.

8 CONDITION OF EQUIDISTANCE

Equidistance refers to the preservation of distances measured in a single direction. This property can only be maintained along certain lines or from specific points. When distances are preserved along the meridians, the projection is called meridionally equidistant; when this preservation occurs along the parallels, the projection is called transversally equidistant (Gaspar, 2005).

More specifically, according to Bugayevskiy and Snyder (1995, p. 52), “In normal cylindrical projections equidistant along the meridians, the ordinate is determined so that the scale along all meridians is preserved”. This condition can be represented mathematically by means of the first Gaussian fundamental quantities directly from Equation (119), since, by setting α = 0°, one obtains the linear scale factor along a meridian, here denoted by m_m, and sets it equal to 1. These considerations can be seen in Equations (141) and (142):

$$m_m^2 = \frac{\bar{E}}{e} cos^2 0° + \frac{\bar{G}}{g} sin^2 0° \tag{141}$$

$$\frac{\bar{E}}{e} = 1 \tag{142}$$

Equation (142) represents the condition of equidistance along the meridians for normal cylindrical projections, in terms of the first Gaussian fundamental quantities (Deakin, 2003). One advantage of this type of projection is that, if one wishes to simultaneously reduce area and shape distortions, the equidistant cylindrical projection would be the best choice (Györffy, 1990).

9 PRACTICAL EXAMPLE

Three well-known and important normal cylindrical projections, namely the Mercator projection, the Lambert cylindrical equal-area projection, and the Plate Carrée projection, may exemplify the three types of distortion analyzed in this paper, thereby highlighting the importance of their study. These projections will be illustrated together with indicatrices ellipses and their respective distortion analyses, as shown in Figures 3, 4, and 5.

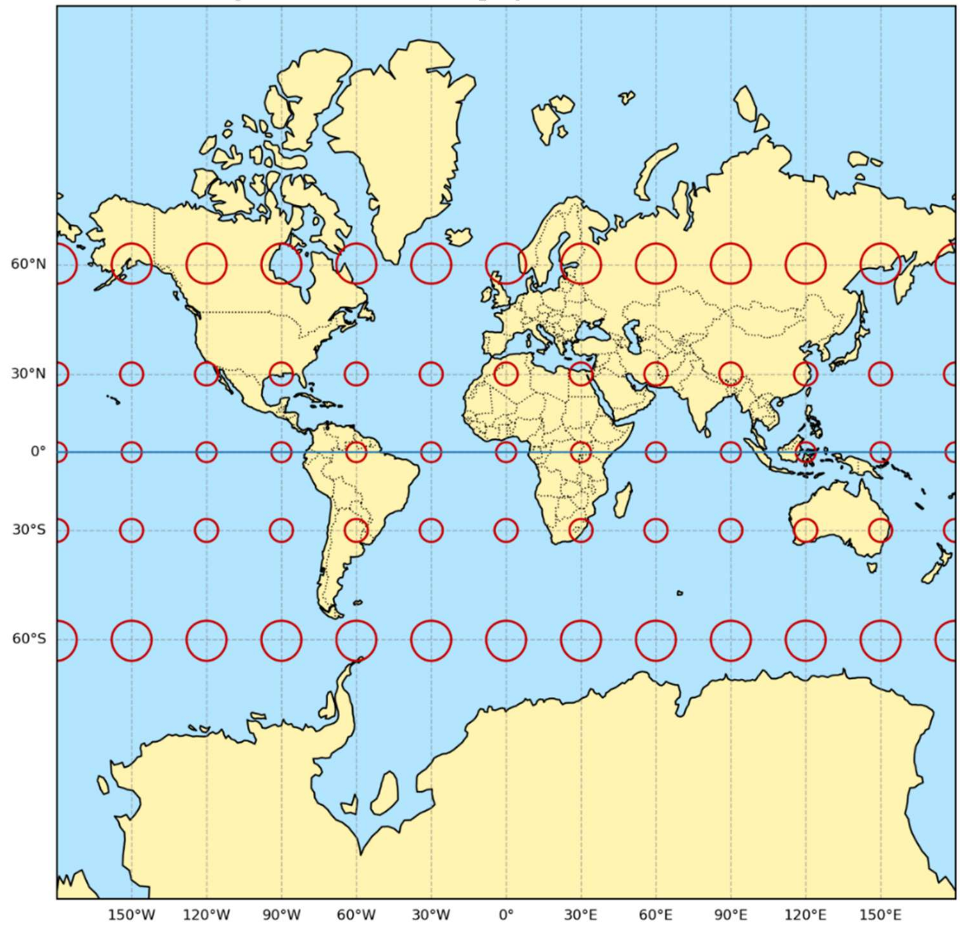
The indicatrix ellipse, proposed by Auguste Tissot in the eighteenth century, is a central concept for graphically understanding deformations in map projections. He showed that any representation of a surface can be regarded as part of several orthogonal projections at different scales. If a small circle on the original surface is taken and projected onto the plane, the process involves two steps: adjusting its scale and performing the orthogonal projection. Since the plane of the circle is generally inclined with respect to the projection plane, its image becomes an ellipse. If the circle has unit radius, the indicatrix ellipse, or ellipse of deformation, is obtained. Its axes a and b correspond to the principal directions of the projection, in which the scale assumes its maximum and minimum values (Gaspar, 2005). These ellipses are important in the study of map projections, both for illustrating distortion and for serving as a basis for the calculations that precisely represent its magnitude at each point (Kopacz, 2012).

The Mercator projection was introduced in 1569 by the cartographer of the same name, primarily for use in navigation. It has the following characteristics: it is a normal cylindrical projection; it is conformal; the meridians are equally spaced straight lines; the parallels are unequally spaced straight lines, closer to one another near the Equator, intersecting the meridians at right angles; the scale is true along the Equator; the poles lie at infinity; and there is great areal distortion in polar regions (Snyder, 1987). Figure 3 illustrates the outline of the continents in the Mercator projection, based on a Python script prepared and made publicly available in a GitHub repository (Data, 2025).

In Figure 3, it can be observed that the indicatrix ellipses retain a circular shape at all latitudes because the projection is conformal, that is, it preserves local shapes. However, the size of the ellipses increases progressively as they move away from the Equator, indicating distortion of area and distance at higher latitudes. Thus, although local shapes are represented correctly, regions farther from the Equator appear disproportionately distorted in relation to those closer to it.

It should be noted that a conformal projection preserves local shapes, since, for example, a small circle on the globe remains circular on the map. However, this property has limitations: conformality requires a uniform increase in scale, causing the areal scale to grow with the square of that variation. Therefore, although small shapes are well represented, the rapid increase in distortions away from zones without deformation makes conformal projections unsuitable for accurately representing large regions, such as continents and oceans (Maling, 1992).

Figure 3 – A Mercator projection visualization.



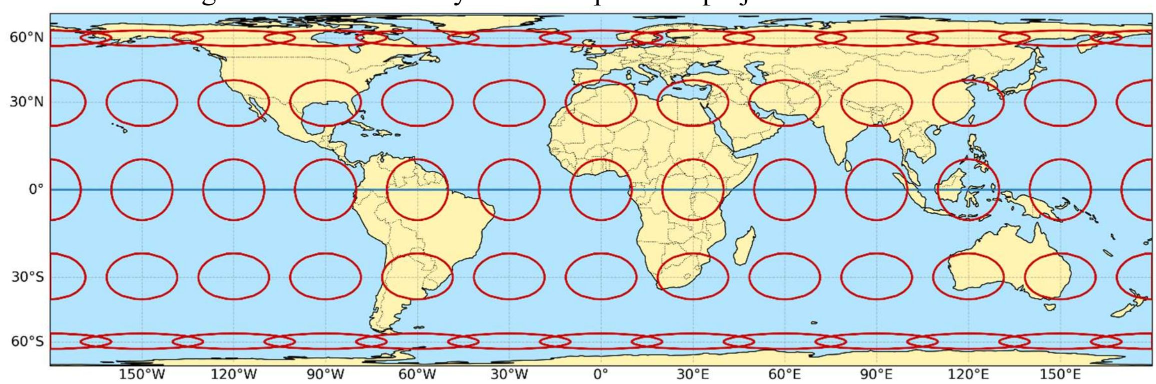
Source: The Authors (2026).

The Lambert cylindrical equal-area projection was introduced by the cartographer of the same name in 1772 and, in its normal aspect, has the following characteristics: the meridians are equally spaced straight lines; the parallels are unequally spaced straight lines, closer to one another near the poles, intersecting the meridians at right angles; and, when tangent, the scale is true along the Equator (Snyder, 1987). Figure 4 illustrates the outline of the continents in this projection, based on a Python script (Data, 2025).

In the Lambert cylindrical equal-area projection, the indicatrix ellipses become increasingly flattened as latitude increases, progressively stretching in the east–west direction. Distortion intensifies in high-latitude regions, where significant deformations of distances and angles occur, although the preservation of equal-area property is maintained.

In general, equal-area projections are highly relevant for mapping statistical distributions. For example, when representing population, agriculture, or industry data, various symbols, such as dots, may be used to indicate specific quantities of the variable under analysis. In reading this type of map, an essential factor is the visual perception of population density, agricultural production, or industrial activity as these vary across different regions of a country or continent (Maling, 1992).

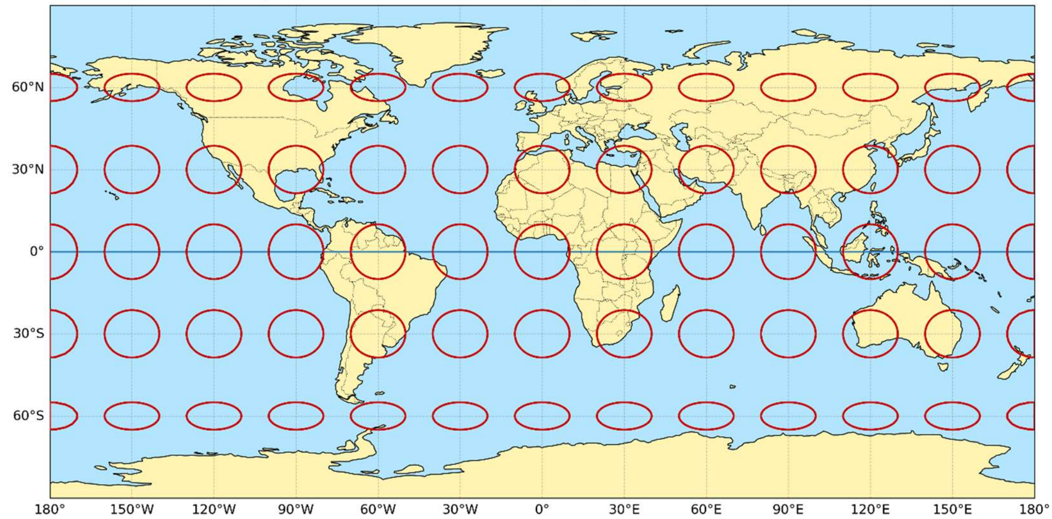
Figure 4 – A Lambert cylindrical equal-area projection visualization.



Source: The Authors (2026).

Equidistant cylindrical projection has been known in Europe since the third century BC. In its equatorial aspect, it is the simplest of all map projections to construct. When the cylinder is in normal position and the Equator is the standard parallel, it is known as Plate Carrée because its graticule is square (Gaspar, 2005). This projection is cylindrical; it is used only in spherical form; it is neither equal-area nor conformal; the meridians and parallels are equally spaced straight lines that intersect at right angles; and it is widely used for world or regional maps because of its simple construction (Snyder, 1987). Figure 5 illustrates the outline of the continents in the Plate Carrée projection, based on a Python script (Data, 2025).

Figure 5 – A Plate Carrée projection visualization.



Source: The Authors (2026).

As shown in Figure 5, the equidistance property of the Plate Carrée projection preserves distances along the meridians. This can be observed through the indicatrix ellipses, which in this case maintain a constant east–west dimension but become compressed in the north–south direction as latitude increases, due to the constant spacing of the parallels. Consequently, at high latitudes, the ellipses appear compressed, revealing severe distortion of shape and area, which reaches its maximum near the poles.

The property of equidistance is less important than conformality or equal-area property, since a map in which distances are exact in only one direction is generally not desirable. Even so, equidistant projections serve as an intermediate solution between conformal and equal-area projections. In these projections, the variation in areal scale occurs more progressively than in conformal projections, and the maximum angular deformation increases more gradually than in equal-area projections. For this reason, they are often used in atlas maps, planning maps, and broad representations of the Earth's surface when neither conformality nor equal-area property needs to be preserved (Maling, 1992).

It should also be emphasized that, in the case of these three projections, there is a direct relationship between the coordinate grid and the distortion pattern. This means that the grid has the same axes of symmetry as the distortion pattern. This occurs because, in the normal aspect, lines of equal distortion coincide with the parallels. In the case of cylindrical projection, the line of zero distortion corresponds to the Equator. Therefore, the normal aspect of this class of projections is known as the equatorial aspect (Canters and Decleir, 1989).

10 FINAL CONSIDERATIONS

This article derived, using the first Gaussian fundamental quantities, the conditions of conformality, equivalence, and equidistance for normal cylindrical projections tangent to the reference surface along the Equator.

Since such projections present constant distortion along parallels, they are a suitable choice for zones that are symmetric with respect to the Equator (Kerkovits, 2021). Moreover, when the cylinder is developed, they produce a map in which the meridians are represented by straight lines parallel to one another, whose spacing is proportional to the corresponding difference in longitude. The parallels are straight lines perpendicular to the meridians and, therefore, are also mutually parallel. However, three types of projections were presented, distinguished by the fact that one is conformal, the Mercator projection; another is equal-area, the Lambert cylindrical equal-area projection; and another is meridionally equidistant, the Plate Carrée projection. These characteristics are guaranteed by the mathematical framework involved, which establishes the conditions required for each property.

As this article has emphasized, an adequate understanding of the statements above depends directly on intermediate to advanced mathematical knowledge. Once acquired, however, such knowledge proves advantageous for those who work directly with map production. This is evident in the practical example, in which three projections were illustrated, together with their indicatrix ellipses, using a Python program whose development required a clear understanding of the concepts involved and of the necessary mathematics, all related to the specific characteristics of each projection.

REFERENCES

Budke, A. Didactic Approaches in Cartography. *KN - Journal of Cartography and Geographic Information* (2025) 75:173–176.

Bugayevskiy, L. M., Snyder, J. P. (1995). *Map Projections, a Reference Manual*. Taylor & Francis Ltd, London.

Canters, F., Decleir, H. (1989). *The World in Perspective, a Directory of World Map Projections*. John Wiley & Sons, New York.

Carmo, M. P. (2016). *Differential Geometry of Curves and Surfaces: Revised & Updated Second Edition.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Data (2025). *Data-cartographic-projections*. GitHub. https://github.com/Data92repository/Data-cartographic-projections.

Deakin, R. E. (2003). *Map Projection Theory*. RMIT University, Melbourne.

Eberth, A., Wagener, J. & Zink, R. Digital Participation Platforms as Tools for Critically Reflexive Mapping in Geography Education: Insights from a Project with Teacher Training Students. *KN J. Cartogr. Geogr. Inf.* 75, 219–231 (2025). https://doi.org/10.1007/s42489-025-00201-4

Edler, D., Wiktorin, D., Becker, G. *et al.* The Potential of Immersive Virtual Reality for Teaching Geography in Schools. *KN J. Cartogr. Geogr. Inf.* 75, 205–217 (2025). https://doi.org/10.1007/s42489-025-00204-1

Ghaderpour, E. (2016). Some Equal-area, Conformal and Conventional Map Projections: A Tutorial Review. *Journal of Applied Geodesy* 2016; 10(3): 197–209.

Gaspar, J. A. (2005). *Cartas e Projeções Cartográficas* 3ª edição. Lidel, edições técnicas Ltda, Lisboa.

Gauss, C. F. (1902). General Investigations of Curved Surfaces. Translated with notes and a bibliography by James Caddall Morehead, and Adam Miller Hiltebeitel. The Princeton University Library.

Györffy, J. (1990). Anmerkungen zur Frage der Besten Echten Zylinderabbildungen. *Kartographische Nachrichten* 40(4), pp. 140–146.

Kerkovits, K. (2021). Best Cylindrical Map Projections According to the Undesirability of Angular and Areal Distortions. *Advances in Cartography and GIScience of the International Cartographic Association*, 3, 2021. 30th International Cartographic Conference (ICC 2021), 14–18 December 2021, Florence, Italy. https://doi.org/10.5194/ica-adv-3-8-2021.

Kessler, F.C. (2025). Map Projections don't Have to be Hard. *Cartographic Perspectives*. Number 106, 2025. DOI: 10.14714/CP106.2029.

Kessler, F. C., Battersby, S. E. (2019). *Working with Map Projections: A Guide to Their Selection*. CRC Press, Boca Raton.

Kopacz, P. (2012). On Geometric Properties of Spherical Conics and Generalization of π in Navigation and Mapping. *Geodesy and Cartography*, 2012 Volume 38(4): 141–151, DOI:10.3846/20296991.2012.756995.

Krakiwsky, E. J. (1973). *Conformal Map Projections in Geodesy*. Technical Report n° 217. Department of Geodesy and Geomatics Engineering, University of New Brunswick, Fredericton.

Kunze, I.S., Budke, A., Krause, U. *et al.* Map-Based Argumentation in Teacher Training: Conceptual Beliefs of Geography Students in Germany, the Netherlands, and Spain. *KN J. Cartogr. Geogr. Inf.* 75, 233–243 (2025). https://doi.org/10.1007/s42489-025-00205-0

Kunze, I.S., Budke, A. Map Use in Geography Lessons: Teachers' Concepts of Map-Based Argumentation. *KN J. Cartogr. Geogr. Inf.* 75, 255–264 (2025). https://doi.org/10.1007/s42489-025-00198-w

Lapaine, M. (2021). Local Linear Scale Factors in Map Projections of an Ellipsoid. *Geographies*, 2021, 1, 238–250. https://doi.org/10.3390/geographies1030014.

Lapaine, M. (2025). A New Derivation of the Formula for the Length of a Loxodrome Arc on a Sphere Using Cylindrical Projections. *ISPRS Int. J. Geo-Inf.* 2025, 14, 137. https://doi.org/10.3390/ijgi14040137.

Maling, D. H. (1992). *Coordinate Systems and Map Projections, second edition*. Pergamon Press, Oxford.

Melluish, R. K. (2014). *An Introduction of Mathematics of Map Projections*. Cambridge University Press, Cambridge: first paperback edition 2014.

Meyer, T. H. (2018). *Introduction to Geometrical and Physical Geodesy: Foundations of Geomatics*. ESRI press, Redlands.

Mocnik, F. B. (2025). Compromise Projections for World Maps – An Optimization Approach Using Discrete Global Grid Systems and Principles from Classical Mechanics. *International Journal of Digital Earth* 2025, Vol. 18, n°. 1, 2369636. https://doi.org/10.1080/17538947.2024.2369636.

Morawski, M., Budke, A. What Is a Map in Digital Games? Expanding Cartographic Literacy Through Ludic Spatial Interfaces in Geography Education. *KN J. Cartogr. Geogr. Inf.* 75, 177–189 (2025). https://doi.org/10.1007/s42489-025-00200-5

Moser, J. Choropleth Maps: Critical Reflections on the Frequently Used Representation Method in an Educational Context. *KN J. Cartogr. Geogr. Inf.* 75, 191–204 (2025). https://doi.org/10.1007/s42489-025-00203-2

Novikova, E. (2025). *Best Map Projections.* Springer Nature Switzerland AG 2025.

Pearson, I. (1990). *Map Projections Theory and Applications*. CRC Press, Boca Raton.

Ramos Junior, I., Seixas, A., Garnés, S. J. A., Calado, L. G. L. P. (2025). Mercator Projection on the Sphere, a Deduction Without Mathematical Gap. *Mercator, Fortaleza*, v. 24, june 2025. https://doi.org/10.4215/rm2025.e24014.

Richardus, P., Adler, R. K. (1972). *Map projections for geodesists, cartographers and geographers*. North-Holland Publishing Company, Amsterdam.

Roblin, H. S. (1969). *Map Projections*. Edward Arnold (Publishers) Limited.

Santos, A. A. (1985). *Representações Cartográficas*. Editora Universitária da Universidade Federal de Pernambuco, Recife.

Snyder, J. P. (1987). *Map Projections, a Working Manual*. United States Geological Survey Professional Paper 1395. United States Government Printing Office, Washington.

Tobler, W. R. (1962). A Classification of Map Projections. *Annals of the Association of American Geographers*, vol 52: p-p 167-175.

Ulbricht, J., Lindau, AK. Pre-Service Geography Teachers' Spatial Conceptions—How Do Field Trips Shape Spatial Conceptions?. *KN J. Cartogr. Geogr. Inf.* 75, 245–254 (2025). https://doi.org/10.1007/s42489-025-00208-x

Contribuição de autoria – Authors' contributions

Isaac Ramos: Conceptualization, Formal Analysis, Investigation, Methodology, Project Administration, Software, Supervision, Validation, Visualization, Writing – Original Draft Preparation, Writing – Review & Editing; Leonard Silveira: Conceptualization, Formal Analysis, Investigation, Methodology; Vinícius Martins: Formal Analysis, Investigation, Methodology, Software, Validation, Visualization; Robert Silva: Investigation, Supervision, Writing – Original Draft Preparation, Writing – Review & Editing; Andréa Seixas: Conceptualization, Formal Analysis, Investigation, Methodology, Writing – Original Draft Preparation, Writing – Review & Editing; Sílvio Garnés: Formal Analysis, Methodology, Writing – Review & Editing; Lucas Calado: Conceptualization, Formal Analysis, Investigation, Methodology, Writing – Original Draft Preparation, Writing – Review & Editing; Paulo Airoldi: Methodology, Writing – Review & Editing.

Disponibilidade de Dados – Data Availability

The entire dataset supporting the results of this study has been made available in Github and can be accessed at https://github.com/data92repository/data-cartographic-projections

Conflito de Interesses – Conflict of Interest

The authors declare that they have no financial, personal, professional, or institutional conflicts of interest that could have influenced, or could be perceived as having influenced, the conduct of the research, the analysis and interpretation of the data, the writing of the manuscript, or the decision to submit it for publication. All authors approved the final version of the manuscript and agree with its submission for publication.