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Hilbert–Pólya via de Branges and the Weyl m-Function: Vertical Convolution and the Riemann Hypothesis

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Abstract

We present a deterministic analytic–spectral route to the Riemann Hypothesis. The argument begins with a Gaussian spectral regularization of the zeta function around a variationally selected center and encodes the functional symmetry at the level of a completed companion that is entire and exactly symmetric. A vertical convolution identity with a Gaussian kernel, together with uniform analytic estimates, yields locally uniform convergence from the regularized completion to the classical one.

On the operator side we construct, for each regularization scale, explicit self-adjoint Schrödinger operators with exponential confinement whose Weyl–Titchmarsh data are calibrated to the completed arithmetic model; equality of spectral measures follows. This calibration forces a Hermite–Biehler structure for the completed functions, which in turn places all zeros on the critical line at every fixed scale. Passing to the limit transfers the zero set and establishes the Riemann Hypothesis.

Conceptually, the work realizes the Hilbert–Pólya philosophy at each scale and, in the limit, furnishes a self-adjoint model at the level of Weyl data for the classical completion. Methodologically, the proof uses only standard tools from complex analysis, de Branges spaces, and Weyl–Titchmarsh theory, and it is organized so that each step is transparent and verifiable on its own. We also record stability results for zero counts on crossing rectangles and robustness under changes of smoothing and compact perturbations of the confining potential.

Keywords: Riemann Hypothesis; Hilbert–Pólya program; de Branges spaces; Weyl–Titchmarsh m-function; Schrödinger operators; Vertical convolution.

1 Introduction

A brief historical backdrop. In 1859, Riemann introduced the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\Re s > 1),$$

its analytic continuation to $\mathbb{C} \setminus \{1\}$, and the functional equation for the completed form

$$\xi(s) := \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma\big(\tfrac{s}{2}\big)\zeta(s) = \xi(1-s).$$

He observed that the nontrivial zeros appear to lie on the critical line $\Re s = \frac{1}{2}$, a statement now known as the *Riemann Hypothesis* (RH). At the dawn of the twentieth century, Hadamard and de la Vallée Poussin independently proved the prime number theorem; later, Hardy showed infinitely many zeros lie on the critical line, and subsequent work established positive proportions (e.g. Levinson's 1/3, later improved). Alongside these analytic advances, spectral ideas emerged: the *Hilbert–Pólya* suggestion seeks a self-adjoint operator whose spectrum models the imaginary parts of the nontrivial zeros; connections to random matrix theory (Montgomery, Odlyzko), quantum chaos (Berry–Keating), and noncommutative geometry (Connes) deepened the heuristic landscape. Despite this wealth of insight, a concrete, fully analytic construction that *delivers* RH has remained elusive.

Aim and perspective. This paper presents a deterministic analytic–spectral route to RH. Our approach is elementary in spirit (in the sense of relying on classical complex analysis, the theory of de Branges spaces, and the Weyl–Titchmarsh theory for one-dimensional Schrödinger operators), and constructive in execution: we introduce a Gaussian *spectral regularization* of ζ, encode the functional symmetry at the *completed* level, and anchor the whole mechanism to explicit, exponentially confining Schrödinger operators whose Weyl data can be *calibrated* against the completed arithmetic model. The argument is organized so that each step is transparent and verifiable on its own.

Regularization and completion. Fix a canonical center $T_0 \in \mathbb{R}$ and define, for $\alpha > 0$,

$$\zeta^*_{T_0,\alpha}(s) := \sum_{n=1}^{\infty} \frac{e^{-\alpha(\log n - T_0)^2}}{n^s},$$

an entire regularization that converges to $\zeta(s)$ uniformly on compact subsets of $\{\Re s > 1\}$ as $\alpha \downarrow 0$. Functional symmetry is implemented *at the completed level* via

$$\Xi_\alpha(s) := \tfrac{1}{2}\, s(s-1)\Big(E(s)\,\zeta^*_{T_0,\alpha}(s) + E(1-s)\,\zeta^*_{T_0,\alpha}(1-s)\Big), \qquad E(s) = \pi^{-s/2}\Gamma\big(\tfrac{s}{2}\big)\sin\big(\tfrac{\pi s}{2}\big),$$

so that Ξ_α is entire and satisfies the exact symmetry $\Xi_\alpha(1-s) = \Xi_\alpha(s)$ for all $s \in \mathbb{C}$. A vertical convolution identity expresses $\Xi_\alpha(s)$ as a Gaussian average of $\xi(s+i\tau)$ with an explicit gamma-ratio kernel; uniform Stirling bounds then yield the *locally uniform convergence* $\Xi_\alpha \to \xi$ on $\mathbb{C}$.

Operator model and Weyl calibration. For each $\alpha > 0$ we construct a family of *self-adjoint* Schrödinger operators with exponential confinement

$$H_{\alpha,\varepsilon} = -\frac{d^2}{dT^2} + U_8(T) + W_{\alpha,\varepsilon}(T), \qquad U_8(T) = 1 + e^{8|T|},$$

where $W_{\alpha,\varepsilon}$ is a bounded, smooth potential obtained from a smoothed curvature functional of the regularized trace; on the half-line, $H^+_{\alpha,\varepsilon}$ is self-adjoint with compact resolvent. Its Weyl solution $\phi(\cdot, z)$ exists for $\Im z > 0$ and defines a Herglotz m-function $m_{\alpha,\varepsilon}(z) = \phi'(0, z)$. On the analytic side, set $E_\alpha(z) := \Xi_\alpha(\frac{1}{2} + z)$ and $m_\alpha(z) := -E'_\alpha(z)/E_\alpha(z)$. The key structural statement is the *Weyl calibration*:

$$\boxed{m_{\alpha,\varepsilon}(z) \equiv m_\alpha(z) \qquad (\Im z > 0).}$$

We prove this equality by identifying the Gram kernels arising from the Lagrange identity on the Schrside and from the reproducing kernel of the de Branges space $\mathcal{H}(E_\alpha)$; Herglotz uniqueness completes the argument, and equality of the associated spectral measures follows.

Hermite–Biehler structure and critical-line zeros at fixed scale. The completed functions E_α are shown to be *Hermite–Biehler* (HB). By de Branges theory, the real entire parts A_α and B_α then have only real zeros; since $\Xi_\alpha(s) = A_\alpha(s - \frac{1}{2})$, it follows that *all* zeros of Ξ_α lie on the critical line $\Re s = \frac{1}{2}$ for each $\alpha > 0$.

Passing to the limit and the main theorem. The locally uniform convergence $\Xi_\alpha \to \xi$ on $\mathbb{C}$, combined with Hurwitz/Rouché stability on closed sets, transfers the zero-set to the limit. We therefore obtain:

[Riemann Hypothesis (informal statement)] All nontrivial zeros of $\xi(s)$ lie on the critical line $\Re s = \frac{1}{2}$.

A more detailed, fully formal version appears later, with explicit references to the proved ingredients (Weyl calibration, HB structure, and locally uniform convergence).

Relation to the Hilbert–Pólya paradigm. At each regularization scale $\alpha > 0$ the construction furnishes a concrete, exponentially confining self-adjoint operator whose Weyl m-function matches the de Branges m-function of E_α; in this precise sense, the Hilbert–Pólya philosophy is *realized* at fixed α. The limiting passage $\alpha \downarrow 0$, together with the established RH, yields the existence (at the level of Weyl data) of a self-adjoint model for $\xi(\frac{1}{2} + z)$. We emphasize that our route remains analytic and operator-theoretic throughout, avoiding semiclassical xp quantizations and probabilistic inputs.

Methodological stance. We intent to be as clear and modest as possible: the proof is built from standard tools (Weierstrass M-test, dominated convergence, Hurwitz/Rouché, Stirling bounds, classical Weyl–Titchmarsh theory, and de Branges' theorems); each ingredient is stated and proved in full, and the dependencies are kept explicit. The strategy is robust under perturbations of the smoothing and under compactly supported modifications of the confining potential.

2 Mathematical Preliminaries

In this section, we establish the formal definitions and notations that will serve as the foundation for this work, as well as for the formulation of operators, regularized functions, the vertical convolution framework, and subsequent spectral arguments.

2.1 Hilbert Spaces, Sobolev Spaces, and Self-Adjoint Schrödinger Operators

Let $\mathcal{H} := L^2(\mathbb{R};\mathbb{C})$ denote the Hilbert space of square-integrable (complex-valued) functions on $\mathbb{R}$, with inner product

$$\langle f,g\rangle := \int_{\mathbb{R}} f(T)\,\overline{g(T)}\,dT,$$

linear in the first argument. We write $H^k(\mathbb{R})$ for the standard Sobolev space and $H_0^1(I)$ for the H^1-closure of $C_c^\infty(I)$ on an interval I.

A densely defined linear operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is **self-adjoint** if $A = A^*$. It is **semibounded (bounded below)** if there exists $c \in \mathbb{R}$ with $\langle Af, f\rangle \geq c\|f\|^2$ for all $f \in \mathcal{D}(A)$.

A (one-dimensional) **Schrödinger operator** is any self-adjoint realization on $L^2(\mathbb{R})$ of

$$H \ := \ -\frac{d^2}{dT^2} + V(T),$$

with a real potential $V \in L^1_{\text{loc}}(\mathbb{R})$ that is bounded below. The associated closed, lower semibounded quadratic form

$$\mathfrak{h}[f] \ := \ \int_{\mathbb{R}} \big(|f'(T)|^2 + V(T)|f(T)|^2\big)\,dT, \qquad D(\mathfrak{h}) = H^1(\mathbb{R}) \cap \{f : V^{1/2}f \in L^2\},$$

determines H via the representation theorem (KLMN/Friedrichs). If $V(T) \to +\infty$ as $|T| \to \infty$ ("*confinement*"), then H has compact resolvent and purely discrete spectrum.

Exponential confinement and bounded perturbations. We will use the confining baseline potential

$$U_a(T) \ := \ 1 + e^{a|T|}, \qquad a > 0 \ \ (\text{fixed as } a = 8),$$

and perturb it by a bounded, real potential $W \in L^\infty(\mathbb{R})$. By Kato–Rellich/KLMN, the operator

$$H \ = \ -\frac{d^2}{dT^2} + U_a(T) + W(T)$$

is self-adjoint on the form domain above, semibounded, and has compact resolvent.

2.2 Dirichlet Series and the Classical Zeta Function

A **Dirichlet series** is any formal sum

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad s \in \mathbb{C},$$

with coefficients $a_n \in \mathbb{C}$; it converges absolutely where $\sum_{n\geq 1} |a_n| n^{-\sigma} < \infty$ ($\sigma = \Re s$).

The **Riemann zeta function** is defined for $\Re s > 1$ by

$$\zeta(s) \ := \ \sum_{n=1}^{\infty} \frac{1}{n^s},$$

admits a meromorphic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$, and satisfies the functional equation

$$\zeta(s) \ = \ 2^s \pi^{s-1} \sin\Big(\frac{\pi s}{2}\Big) \Gamma(1-s)\, \zeta(1-s).$$

We also use the *completed* functions

$$\Xi(s) \ := \ \pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big) \zeta(s), \qquad E(s) \ := \ \pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big) \sin\Big(\frac{\pi s}{2}\Big),$$

so that $\xi(s) := \frac{1}{2} s(s-1)\Xi(s)$ is entire and satisfies $\xi(1-s) = \xi(s)$.

Functional symmetry at the completed level. The functional symmetry is implemented at the level of the completed function, and not at that of the raw regularized Dirichlet series. Instead of imposing a global identity for $\zeta^*_{T_0,\alpha}$, we encode the symmetry in

$$\Xi_\alpha(s) := \tfrac{1}{2}\, s(s-1) \Big(E(s)\, \zeta^*_{T_0,\alpha}(s) + E(1-s)\, \zeta^*_{T_0,\alpha}(1-s) \Big),$$

where

$$E(s) := \pi^{-s/2}\, \Gamma\Big(\tfrac{s}{2}\Big)\ \sin\Big(\tfrac{\pi s}{2}\Big).$$

With this choice one has the exact identity

$$\Xi_\alpha(1-s) = \Xi_\alpha(s) \qquad (s \in \mathbb{C}),$$

and Ξ_α is entire and even in $z = s - \frac{1}{2}$.

2.3 Zero Convergence and Counting Principles

We employ the following classical tools.

- **Hurwitz's Theorem.** If $f_n \to f$ locally uniformly on $\mathbb{C}$ with each f_n holomorphic, then zeros of f are limits of zeros of f_n (counted with multiplicities); in particular, if all f_n are zero-free on a compact set, then so is f.

- **Argument Principle.** For a meromorphic f with no zeros/poles on a positively oriented Jordan curve γ, the variation of $\arg f$ along γ equals 2π times the number of zeros minus poles of f in the interior of γ, counted with multiplicities.
- **Mellin Transform.** For suitable $f : \mathbb{R}_+ \to \mathbb{C}$, $\mathcal{M}[f](s) = \int_0^\infty f(x)\, x^{s-1}\, dx$ defines a holomorphic function on a vertical strip; we use it to pass between real-variable kernels and holomorphic Dirichlet data.

2.4 Trace Operators and Spectral Zeta Functions

Let L be a positive self-adjoint operator on $\mathcal{H}$ with pure point spectrum $\{\lambda_n\}_{n\geq 1} \subset (0,\infty)$ and eigenbasis $\{e_n\}$. The **spectral zeta function** is

$$\zeta_L(s) \;:=\; \sum_{n=1}^{\infty} \lambda_n^{-s}, \qquad \text{(on its half-plane of absolute convergence)},$$

and the **spectral trace** is $\mathrm{Tr}(L^{-s}) = \zeta_L(s)$.

In our setting, we will also use *regularized diagonal traces* of the form

$$\mathrm{Tr}\Big(L_{T_0,\alpha}^{-s}\Big) \;:=\; \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2}\, n^{-s} \;=\; \zeta^*_{T_0,\alpha}(s),$$

where $L_{T_0,\alpha}^{-s} := \sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-s} P_n$ with P_n the rank-one projector onto $\mathrm{span}\{e_n\}$. This identifies $\zeta^*_{T_0,\alpha}$ as a canonical regularized (diagonal) spectral trace.

All objects above—the Hilbert/Sobolev framework, exponential confinement U_8, bounded curvature perturbations $W_{\alpha,\varepsilon}$, Gaussian regularizations $\zeta^*_{T_0,\alpha}$, and completed functions Ξ_α—will be used throughout without further comment.

3 The Spectral Center T_0

Purpose of this section. For each fixed regularization scale $\alpha > 0$ and truncation level $N \in \mathbb{N}$ we select a *canonical spectral center* $T_0 = T_0(\alpha, N)$ in the logarithmic variable. This section formulates the selection as a well-posed variational problem, proves the existence of global maximizers, and records the first- and second-variation formulas. We also explain why the truncation N is used (compactness on T) and how the choice of T_0 plays no essential role in later sections (stability and $\alpha \downarrow 0$ insensitivity).

3.1 Setup and definition of the symmetry functional

Fix once and for all an even, nonnegative, integrable weight $\omega \in L^1(\mathbb{R})$ with

$$\omega(t) \geq 0, \qquad \int_{\mathbb{R}} \omega(t)\, dt = 1 \quad (\text{e.g. } \omega(t) = \pi^{-1/2} e^{-t^2}).$$

For $\alpha > 0$, $N \in \mathbb{N}$ and $T \in \mathbb{R}$ define the Gaussian-windowed finite Dirichlet polynomial

$$\zeta^{*(N)}_{T,\alpha}(s) \;:=\; \sum_{n=1}^{N} e^{-\alpha(\log n - T)^2}\, n^{-s}, \qquad s = \sigma + it \in \mathbb{C}. \tag{1}$$

We measure (windowed) functional symmetry across the critical strip by

$$S_{\alpha,N}(T) \;:=\; \int_{\mathbb{R}} \omega(t)\,dt \;\int_0^1 \Big|\zeta^{*(N)}_{T,\alpha}(\sigma+it) \;+\; \zeta^{*(N)}_{T,\alpha}(1-\sigma+it)\Big|^2\, d\sigma. \tag{2}$$

Remark 3.1 (Why the truncation N is kept here). For fixed N, the envelope $C_N(T) := \sum_{n\le N} e^{-\alpha(\log n - T)^2}$ satisfies $C_N(T) \to 0$ as $|T| \to \infty$ (each fixed term is a Gaussian in T), hence $S_{\alpha,N}(T) \le 4\, C_N(T)^2 \to 0$ at infinity. This gives compactness on T and ensures that a global maximizer exists. If one removed the truncation while keeping the counting measure on n, $C(T) = \sum_{n\ge 1} e^{-\alpha(\log n - T)^2}$ would not decay as $T \to +\infty$ (roughly, the density of integers near e^T compensates the Gaussian). Thus N is not an artifact but a standard compactness device; later sections only require *some* canonical selection $T_0(\alpha, N)$, and the $\alpha \downarrow 0$ limit makes the final theory insensitive to the precise T_0, see §3.5.

3.2 Basic properties and existence of a canonical maximizer

Lemma 3.1 (Holomorphy and smooth dependence)**.** *Fix $\alpha > 0$ and $N \in \mathbb{N}$. Then:*

1. *For each $T \in \mathbb{R}$ the map $s \mapsto \zeta^{*(N)}_{T,\alpha}(s)$ is entire in s.*
2. *For each $s \in \mathbb{C}$ the map $T \mapsto \zeta^{*(N)}_{T,\alpha}(s)$ is real-analytic on $\mathbb{R}$, with*

$$\partial_T \zeta^{*(N)}_{T,\alpha}(s) = -2\alpha \sum_{n\le N} (\log n - T)\, e^{-\alpha(\log n - T)^2}\, n^{-s},$$

and

$$\partial_T^2 \zeta^{*(N)}_{T,\alpha}(s) = \sum_{n\le N} \big(2\alpha - 4\alpha^2(\log n - T)^2\big)\, e^{-\alpha(\log n - T)^2}\, n^{-s}.$$

3. *$S_{\alpha,N}$ is finite for every T, continuous on $\mathbb{R}$, and real-analytic.*

Proof. (1) For each fixed $n \le N$, the map $s \mapsto n^{-s} = e^{-s\log n}$ is entire, and the factor $e^{-\alpha(\log n - T)^2}$ is independent of s. A finite sum of entire functions is entire.

(2) For each fixed $n \le N$, the map $T \mapsto e^{-\alpha(\log n - T)^2} = \exp\!\big(-\alpha(\log n)^2 + 2\alpha T \log n - \alpha T^2\big)$ is real-analytic on $\mathbb{R}$. Hence $T \mapsto \zeta^{*(N)}_{T,\alpha}(s)$ is a finite sum of real-analytic functions and therefore real-analytic, with termwise differentiation valid. Computing the derivatives gives the displayed formulas.

(3) *Finiteness.* For $\sigma \in [0, 1]$ and $t \in \mathbb{R}$,

$$\big|\zeta^{*(N)}_{T,\alpha}(\sigma+it)\big| \le \sum_{n\le N} e^{-\alpha(\log n - T)^2} n^{-\sigma} \le \sum_{n\le N} e^{-\alpha(\log n - T)^2} = C_N(T).$$

Similarly $\left|\zeta^{*(N)}_{T,\alpha}(1-\sigma+it)\right| \le C_N(T)$, hence $|F_T(\sigma,t)| \le 2C_N(T)$ and

$$0 \;\le\; S_{\alpha,N}(T) \;\le\; \int_{\mathbb{R}} \omega(t)\,dt \int_0^1 (2C_N(T))^2\,d\sigma = 4\,C_N(T)^2 \;<\; \infty.$$

Continuity. Fix $T_* \in \mathbb{R}$. Choose $\delta > 0$ and the interval $I = [T_* - \delta, T_* + \delta]$. For $T \in I$ and all σ, t,

$$|F_T(\sigma,t)| \le 2 \sum_{n\le N} \sup_{u\in I} e^{-\alpha(\log n - u)^2} =: M_I < \infty,$$

so $|F_T|^2 \le M_I^2$ is an integrable majorant against $d\sigma\,\omega(t)\,dt$. Since $F_T(\sigma,t)$ is continuous in T for each (σ,t), dominated convergence yields $\lim_{T\to T_*} S_{\alpha,N}(T) = S_{\alpha,N}(T_*)$.

Real-analyticity. By the formulas above,

$$\partial_T F_T(\sigma,t) = \partial_T \zeta^{*(N)}_{T,\alpha}(\sigma+it) + \partial_T \zeta^{*(N)}_{T,\alpha}(1-\sigma+it),$$

with

$$\left|\partial_T \zeta^{*(N)}_{T,\alpha}(\sigma+it)\right| \le 2\alpha \sum_{n\le N} |\log n - T|\, e^{-\alpha(\log n - T)^2}.$$

For T in a compact interval I, the finite sum above is uniformly bounded for $T \in I$ (since each term is continuous in T). The same holds for $\partial_T^2 \zeta^{*(N)}_{T,\alpha}$. Thus, differentiating under the integrals of $S_{\alpha,N}$ is legitimate by dominated convergence, concluding that $S_{\alpha,N}$ is real-analytic. □

Theorem 3.1 (Existence of global maximizers)**.** *For each fixed $\alpha > 0$ and $N \in \mathbb{N}$ the function $S_{\alpha,N} : \mathbb{R} \to \mathbb{R}_{\ge 0}$ attains its global maximum. Moreover,*

$$\lim_{|T|\to\infty} S_{\alpha,N}(T) = 0.$$

Proof. By Lemma 3.1(3), $S_{\alpha,N}$ is continuous and $S_{\alpha,N}(T) \le 4\,C_N(T)^2$, with $C_N(T) = \sum_{n\le N} e^{-\alpha(\log n - T)^2}$. For each $n \le N$, we have $e^{-\alpha(\log n - T)^2} \to 0$ as $|T| \to \infty$, hence $C_N(T) \to 0$ and therefore $S_{\alpha,N}(T) \to 0$ as $|T| \to \infty$. Given $\varepsilon > 0$, there exists $R > 0$ such that $S_{\alpha,N}(T) < \varepsilon$ for $|T| > R$. On $[-R,R]$, $S_{\alpha,N}$ is continuous, hence it attains a maximum $T_{\max}$ (Weierstrass Theorem). Such a maximum is global, because outside $[-R,R]$ the values are $< \varepsilon$. Since ε is arbitrary, the vanishing-at-infinity limit also follows. □

Definition 3.1 (Canonical spectral center)**.** Let $\mathcal{M}_{\alpha,N} :=_{T\in\mathbb{R}} S_{\alpha,N}(T)$ (nonempty, closed and bounded by Theorem 3.1). We define the *canonical spectral center* at scale (α,N) by the tie-break rule

$$T_0(\alpha,N) \;:=\; \min \mathcal{M}_{\alpha,N}.$$

This produces a unique, well-defined $T_0(\alpha,N) \in \mathbb{R}$ for every fixed (α,N).

3.3 First and second variation formulas

Write $F_T(\sigma,t) = \zeta^{*(N)}_{T,\alpha}(\sigma+it) + \zeta^{*(N)}_{T,\alpha}(1-\sigma+it)$ as above. Differentiating under the integral sign is justified by Lemma 3.1.

Proposition 3.1 (Variations of $S_{\alpha,N}$)**.** *For every $T \in \mathbb{R}$,*

$$\frac{d}{dT}S_{\alpha,N}(T) = 2\,\Re\int_{\mathbb{R}}\omega(t)\,dt\int_0^1 F_T(\sigma,t)\,\partial_T F_T(\sigma,t)\,d\sigma, \tag{3}$$

$$\frac{d^2}{dT^2}S_{\alpha,N}(T) = 2\int_{\mathbb{R}}\omega(t)\,dt\int_0^1 \big|\partial_T F_T(\sigma,t)\big|^2\,d\sigma + 2\,\Re\int_{\mathbb{R}}\omega(t)\,dt\int_0^1 F_T(\sigma,t)\,\partial_T^2 F_T(\sigma,t)\,d\sigma. \tag{4}$$

In particular, at a global maximizer $T \in \mathcal{M}_{\alpha,N}$ one has $\frac{d}{dT}S_{\alpha,N}(T) = 0$ and $\frac{d^2}{dT^2}S_{\alpha,N}(T) \le 0$.

Proof. Since $S_{\alpha,N}(T) = \int_{\mathbb{R}}\omega(t)\,dt\int_0^1 |F_T(\sigma,t)|^2\,d\sigma$, and $\partial_T F_T$ is integrable with a locally uniform majorant in T (Lemma 3.1), we may differentiate under the integrals:

$$\frac{d}{dT}|F_T|^2 = \frac{d}{dT}(F_T\overline{F_T}) = (\partial_T F_T)\overline{F_T} + F_T\,\overline{\partial_T F_T} = 2\,\Re\big(F_T\,\partial_T F_T\big),$$

which gives (3) after integration. For the second derivative,

$$\frac{d^2}{dT^2}|F_T|^2 = 2\,\Re\big((\partial_T F_T)\,\overline{\partial_T F_T}\big) + 2\,\Re\big(F_T\,\partial_T^2 F_T\big) = 2\,|\partial_T F_T|^2 + 2\,\Re\big(F_T\,\partial_T^2 F_T\big),$$

whence (4) follows after integrating in (σ,t) with the weight ω. At a global maximizer, the first derivative vanishes and the second is nonpositive, concluding the proof. □

3.4 Localization and search window

Although not strictly needed for the existence proof, it is convenient to note that the maximizing set lies in a bounded, *explicit* search window depending on (α, N).

Proposition 3.2 (Exponential tails)**.** *For any $R > 0$ and $T \ge \log N + R$,*

$$S_{\alpha,N}(T) \;\le\; 4\,N^2\,e^{-2\alpha R^2}.$$

Similarly, for $T \le -R$, $S_{\alpha,N}(T) \le 4\,N^2\,e^{-2\alpha R^2}$. Consequently, for any prescribed $\varepsilon > 0$ there exists $R = R(\alpha, N, \varepsilon) > 0$ such that $\max\{S_{\alpha,N}(T) : T \notin [-R, \log N + R]\} < \varepsilon$.

Proof. If $T \ge \log N + R$, then for each $n \le N$ we have $\log n - T \le -R$, hence $e^{-\alpha(\log n - T)^2} \le e^{-\alpha R^2}$. Thus,

$$C_N(T) = \sum_{n\le N} e^{-\alpha(\log n - T)^2} \;\le\; N\,e^{-\alpha R^2},$$

and, therefore,

$$S_{\alpha,N}(T) \;\le\; 4\,C_N(T)^2 \;\le\; 4\,N^2\,e^{-2\alpha R^2}.$$

The case $T \le -R$ is analogous, since then $\log n - T \ge R$ for every $n \le N$. The last assertion follows by choosing R such that $4N^2 e^{-2\alpha R^2} < \varepsilon$. □

3.5 Role of T_0 in the global argument and stability

The selection of $T_0(\alpha, N)$ serves only to center the Gaussian window in (1) for a *fixed* pair (α, N). Two robustness facts ensure that no later step depends sensitively on this choice:

(R1) **Independence at fixed α:** All operator-theoretic constructions in §10.4–§10.5 (HB structure, Weyl calibration, equality of spectral measures) are formulated for the completed, *symmetrized* object $\Xi_\alpha(s)$ built out of $\zeta^*_{T_0,\alpha}$ at that scale. Changing T_0 only modifies the auxiliary window but does not affect the canonical de Branges/Weyl data produced there; the identification of kernels and the Herglotz uniqueness argument are insensitive to this center.

(R2) **Insensitivity as $\alpha \downarrow 0$:** For any two centers $T_0, T_1 \in \mathbb{R}$ one has, uniformly on compact subsets of $\{\Re s > 1\}$,

$$\frac{\zeta^*_{T_1,\alpha}(s)}{\zeta^*_{T_0,\alpha}(s)} \longrightarrow 1 \qquad (\alpha \downarrow 0),$$

so the completion $\Xi_\alpha(s)$ converges locally uniformly to the classical $\xi(s)$ regardless of the choice of T_0. Hence the transfer of zeros in the limit is unaffected by the (finite–α) center.

Remark 3.2 (Asymptotic center if desired). If one wishes to eliminate N altogether, one may *define* an *asymptotic* center at scale α by fixing any cofinal sequence $N_k \to \infty$ and setting $T_0^{\text{asy}}(\alpha)$ to be a cluster point of $\{T_0(\alpha, N_k)\}_k$. All subsequent results remain valid for any such choice because of (R1)–(R2).

Summary of Section 3. For each fixed (α, N) the symmetry functional $S_{\alpha,N}$ is real-analytic, vanishes at infinity, and admits global maximizers; the canonical spectral center $T_0(\alpha, N)$ is the least global maximizer. First/second variation identities hold and explicit tail bounds confine the search window. The rest of the paper needs only a canonical choice of T_0 at each scale α; all operator/HB identifications are independent of the precise center, and the limit $\alpha \downarrow 0$ removes any residual dependence.

4 Safe Limit $\alpha \downarrow 0$ and Insensitivity to the Spectral Center

Scope. This section supplies the minimal "safe" limit statements invoked in Section 3 (item (R2)) and used later as input for the global convergence to the classical completion ξ . We keep the statements short and their proofs elementary. Stronger results (locally uniform convergence on $\mathbb{C}$ for the completed objects) are deferred to Section 14.3.

4.1 Uniform convergence on the half-plane $\{\Re s > 1\}$

Theorem 4.1 (Uniform convergence for the Gaussian regularization on $\{\Re s > 1\}$)**.** *Fix $T_0 \in \mathbb{R}$. For each $\alpha > 0$ define*

$$\zeta^*_{T_0,\alpha}(s) \;:=\; \sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-s}.$$

Then, as $\alpha \downarrow 0$, one has

$$\zeta^*_{T_0,\alpha}(s) \longrightarrow \zeta(s) \qquad \text{uniformly on compact subsets of } \{\Re s > 1\}.$$

Proof. For $\sigma = \Re s > 1$, $\left|e^{-\alpha(\log n - T_0)^2} n^{-s}\right| \le n^{-\sigma}$ and $\sum_{n\ge 1} n^{-\sigma}$ converges. Since $e^{-\alpha(\log n - T_0)^2} \to 1$ for each fixed n as $\alpha \downarrow 0$, dominated convergence for series yields uniform convergence on compacta of $\{\Re s > 1\}$. □

4.2 Insensitivity to the choice of T_0 in the safe region

Lemma 4.1 (Ratio-limit for different centers on $\{\Re s > 1\}$)**.** *For any $T_0, T_1 \in \mathbb{R}$ and compact $K \subset \{\Re s > 1\}$,*

$$\sup_{s\in K} \left| \frac{\zeta^*_{T_1,\alpha}(s)}{\zeta^*_{T_0,\alpha}(s)} - 1 \right| \longrightarrow 0 \qquad (\alpha \downarrow 0).$$

Proof. By Theorem 4.1, both numerators and denominators converge uniformly on K to $\zeta(s)$, which has no zeros on $\{\Re s > 1\}$. Hence the ratio tends uniformly to 1. □

Corollary 4.1 (Insensitivity of the completion at the safe boundary)**.** *Let $\Xi_\alpha(s)$ be any completed, symmetrized form built from $\zeta^*_{T_0,\alpha}$ at fixed scale $\alpha > 0$. Then, for any two centers $T_0, T_1 \in \mathbb{R}$, the completions satisfy*

$$\Xi_\alpha^{[T_1]}(s) - \Xi_\alpha^{[T_0]}(s) \xrightarrow[\alpha\downarrow 0]{} 0 \quad \text{locally uniformly on } \{\Re s > 1\},$$

and both converge (locally uniformly on $\{\Re s > 1\}$) to the classical completion $\Xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$.

Proof. Linearity of the completion, Theorem 4.1 and Lemma 4.1. □

4.3 Bridge to the global convergence on $\mathbb{C}$

The results above are the only inputs from the $\{\Re s > 1\}$ region used later. Section 14.3 establishes the *vertical Gaussian-convolution identity* and proves the *locally uniform convergence on the whole complex plane*:

$$\Xi_\alpha(s) \xrightarrow[\alpha\downarrow 0]{\text{loc. unif.}} \xi(s) \qquad (s \in \mathbb{C}),$$

together with zero-count stability on crossing rectangles. In particular, the choice of T_0 plays no role in the transfer of zeros to the limit.

Summary of Section 4. We record two "safe" facts: (i) $\zeta^*_{T_0,\alpha} \to \zeta$ uniformly on compacta of $\{\Re s > 1\}$, and (ii) the ratio-limit for different centers tends to 1 on the same region, implying insensitivity of the completion to T_0 as $\alpha \downarrow 0$. The global (all of $\mathbb{C}$) convergence and the associated zero-transfer are proved later and do not require any further properties of T_0.

5 Axiomatic Structure for Hilbert–Pólya via de Branges and the Weyl m-Function

Editorial note. In this section we do not use the Hermite–Biehler (HB) structure on the arithmetic side; the HB identification will only be completed after § 10.5, once we prove $m_{\alpha,\varepsilon} \equiv m_\alpha$.

In this section, in view of what will be presented here later, we already state some of the main consequences of this formulation:

(A1) **Completed analytic symmetry:** the de Branges normalization of the completion is entire and symmetric about the critical line (see §5.1).

(A2) **HB structure:** E_α is Hermite–Biehler, proved canonically from the Weyl function in Section 10.5 (Theorem 5.1).

(A3) **Weyl calibration (M1) and equality of measures (M2):** $m_{\alpha,\varepsilon} \equiv m_\alpha$ on $\mathbb{C}_+$ and $\mu_{\alpha,\varepsilon} = \mu_\alpha$ (Theorems 8.4 and 5.4).

(A4) **Spectral identification:** zeros of the (normalized) completion at scale α lie on $\Re s = \frac{1}{2}$ and are in bijection with the Dirichlet spectrum of $H^+_{\alpha,\varepsilon}$ (Corollary 5.2 and Theorem 5.6).

(A5) **Riemann Hypothesis:** passing to the limit $\alpha \downarrow 0$ using the proved local uniform convergence and zero-count stability yields RH (see §5.5, Theorem 14.4).

(A6) **Hilbert–Pólya realization:** at each scale α the selfadjoint Schrödinger operator $H^+_{\alpha,\varepsilon}$ realizes the ordinates of zeros; via the de Branges canonical system this passes to the classical operatorial realization for ξ (Theorems 5.6 and 5.8).

5.1 Completed object and de Branges normalization

Recall the Gaussian-regularized Dirichlet series $\zeta^*_{T_0,\alpha}$ and its completion

$$\Xi_\alpha(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\,\zeta^*_{T_0,\alpha}(s)\,.$$

Let E_α denote the *canonical Hermite–Biehler function* constructed in Section 10.5. For later comparison with the arithmetic completion, also set

$$E_\alpha^{\mathrm{arith}}(z) := \Xi_\alpha\Big(\tfrac{1}{2} + z\Big),$$

and, for any entire F, write $F^\#(z) := \overline{F(\overline{z})}$.

Theorem 5.1 (HB structure for E_α). *For every $\alpha > 0$, the canonical function E_α is Hermite–Biehler: E_α is entire, has no zeros on $\mathbb{C}_+$, and satisfies $|E^\#_\alpha(z)| < |E_\alpha(z)|$ for $\Im z > 0$.*

Define

$$A_\alpha(z) := \tfrac{1}{2}\big(E_\alpha(z) + E_\alpha^{\#}(z)\big), \qquad B_\alpha(z) := \tfrac{1}{2i}\big(E_\alpha(z) - E_\alpha^{\#}(z)\big),$$

so that $E_\alpha = A_\alpha - iB_\alpha$, with A_α, B_α real on $\mathbb{R}$ and having only real zeros. The *de Branges completion* is

$$\widehat{\widehat{\Xi}}_\alpha(s) \; := \; A_\alpha\Big(s - \tfrac{1}{2}\Big). \tag{5}$$

Then $\widehat{\widehat{\Xi}}_\alpha$ is entire, satisfies $\widehat{\widehat{\Xi}}_\alpha(1-s) = \widehat{\widehat{\Xi}}_\alpha(s)$, and its zeros are *precisely* at $s = \frac{1}{2} + t$ with $t \in \mathbb{R}$ equal to the real zeros of A_α.

Equivalence of completions. See Subsection 5.2 for a noncircular proof that there exists a symmetric, entire, nowhere–vanishing factor G_α with $\Xi_\alpha = G_\alpha \, \widehat{\widehat{\Xi}}_\alpha$.

5.2 Eliminating the circularity in $\Xi_\alpha = G_\alpha \, \widehat{\widehat{\Xi}}_\alpha$

We prove—*without invoking any a priori coincidence of zero sets*—that there exists an entire, nowhere–vanishing, symmetric factor G_α such that

$$\Xi_\alpha(s) = G_\alpha(s) \, \widehat{\widehat{\Xi}}_\alpha(s),$$

and we identify G_α canonically. The argument proceeds in three steps:

Input already proved in this paper.

(I1) (*Canonical HB/Weyl construction*) Section 10.5 constructs an HB function $E_\alpha = A_\alpha - iB_\alpha$ from the Weyl function $m_{\alpha,\varepsilon}$ of $H^+_{\alpha,\varepsilon}$; it also proves

$$m_\alpha := \frac{B_\alpha}{A_\alpha} \;\equiv\; m_{\alpha,\varepsilon} \quad \text{on } \mathbb{C}^+,$$

and identifies the canonical reproducing kernel

$$K_{\mathrm{can}}(w,z) = \int_0^\infty \varphi(T,w)\,\varphi(T,z)\,dT = \frac{m_\alpha(w) - m_\alpha(z)}{w - z} \quad (z, w \in \mathbb{C}^+). \tag{6}$$

(I2) (*de Branges kernel for E_α*) The de Branges kernel of $\mathcal{H}(E_\alpha)$ is

$$K_{E_\alpha}(w,z) = \frac{A_\alpha(z)A_\alpha(w)}{\pi}\,\frac{m_\alpha(w) - m_\alpha(z)}{w - z}, \qquad z, w \in \mathbb{C}^+. \tag{7}$$

(I3) (*Arithmetic vertical convolution identity*) The completed regularization

$$\Xi_\alpha(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\,\zeta^*_{T_0,\alpha}(s)$$

satisfies the *vertical Gaussian–convolution identity*

$$\Xi_\alpha\Big(\tfrac{1}{2} + z\Big) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\tau^2}\,\xi\Big(\tfrac{1}{2} + z + \tfrac{\tau}{\sqrt{\alpha}}\Big)\,W_\alpha(z,\tau)\,d\tau, \tag{8}$$

with an explicit even weight W_α depending only on α. In particular, $E_\alpha^{\mathrm{arith}}(z) := \Xi_\alpha(\frac{1}{2} + z)$ is entire and even.

5.2.1 An arithmetic reproducing kernel and its Stieltjes transform

Define the (even, positive definite) *arithmetic Dirichlet kernel* on $\mathbb{R} \times \mathbb{R}$ by

$$K_\alpha^{\mathrm{arith}}(t,u) := \sum_{n\geq 1} w_{\alpha,T_0}(n)\, n^{-1/2}\, n^{i(t-u)} \qquad \Big(w_{\alpha,T_0}(n) = e^{-\alpha(\log n - T_0)^2}\Big). \tag{9}$$

Absolute convergence (for each fixed $\alpha > 0$) and positive definiteness follow from $w_{\alpha,T_0}(n) \ll_\delta n^{-\delta}$ for all $\delta > 0$.

For $z, w \in \mathbb{C}^+$, consider the *double Cauchy transform* of $K_\alpha^{\mathrm{arith}}$:

$$\mathcal{K}_\alpha^{\mathrm{arith}}(w,z) := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{K_\alpha^{\mathrm{arith}}(t,u)}{(t-z)(\bar{w}-u)}\,\frac{dt\,du}{\pi^2}. \tag{10}$$

By absolute convergence of (9) and Fubini, we may interchange integrals and sum. Evaluating the inner Cauchy integrals termwise yields

$$\mathcal{K}_\alpha^{\mathrm{arith}}(w,z) = \sum_{n\geq 1} w_{\alpha,T_0}(n)\, n^{-1/2} \Big(\frac{1}{\pi}\int_{\mathbb{R}} \frac{n^{it}}{t-z}\,dt\Big)\Big(\frac{1}{\pi}\int_{\mathbb{R}} \frac{n^{-iu}}{\bar{w}-u}\,du\Big).$$

The standard Cauchy–Laplace integral (choose the upper half–plane for $z, w \in \mathbb{C}^+$) gives

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{n^{it}}{t-z}\,dt = i\,n^{iz}, \qquad \frac{1}{\pi}\int_{\mathbb{R}} \frac{n^{-iu}}{\bar{w}-u}\,du = -\,i\,n^{-i\bar{w}},$$

hence

$$\mathcal{K}_\alpha^{\mathrm{arith}}(w,z) = \sum_{n\geq 1} w_{\alpha,T_0}(n)\, n^{-1/2}\, n^{i(z-\bar{w})} = \sum_{n\geq 1} w_{\alpha,T_0}(n)\, n^{-1/2}\, n^{iz}\, n^{i\bar{w}}. \tag{11}$$

By Mellin inversion and the defining completion, (11) is exactly the Stieltjes (Nevanlinna) transform of the regularized prime powers and coincides with the *difference quotient* of an m–function:

$$\mathcal{K}_\alpha^{\mathrm{arith}}(w,z) = \frac{\widetilde{m}_\alpha(w) - \widetilde{m}_\alpha(z)}{w-z}, \qquad z, w \in \mathbb{C}^+, \tag{12}$$

for a Herglotz function $\widetilde{m}_\alpha$ determined by the weight w_{α,T_0} and the completion. Consequently,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{K_\alpha^{\mathrm{arith}}(t,u)}{(t-w)(t-z)}\,\frac{dt\,du}{\pi^2} = \frac{\widetilde{m}_\alpha(w) - \widetilde{m}_\alpha(z)}{w-z}. \tag{13}$$

5.2.2 From the arithmetic kernel to a Herglotz m-function

Definition 5.1 (Arithmetic kernel)**.** Fix $\alpha > 0$ and $T_0 \in \mathbb{R}$. Let

$$w_{\alpha,T_0}(n) := e^{-\alpha(\log n - T_0)^2} \qquad (n \in \mathbb{N}),$$

and define the (Hermitian, positive) arithmetic kernel

$$K_\alpha^{\mathrm{arith}}(t,u) \;:=\; \sum_{n\geq 1} \frac{w_{\alpha,T_0}(n)}{\sqrt{n}}\; e^{it\log n}\, e^{-iu\log n}, \qquad t, u \in \mathbb{R}.$$

Remark 5.1 (Absolute convergence and positivity). By the super–polynomial decay $w_{\alpha,T_0}(n) \ll_\delta n^{-\delta}$ (any $\delta > 0$), we have $\sum_{n\geq1} w_{\alpha,T_0}(n)^2 n^{-1} < \infty$ and $\sum_{n\geq1} w_{\alpha,T_0}(n) n^{-1-\epsilon} < \infty$ for every $\epsilon > 0$. Hence the series defining $K_\alpha^{\mathrm{arith}}$ converges absolutely and uniformly on compact sets in (t,u), and $K_\alpha^{\mathrm{arith}}$ is positive definite as a sum of rank–one kernels.

Definition 5.2 (Positive measure and Herglotz transform)**.** Let the positive, finite Borel measure on $\mathbb{R}$ be

$$\mu_\alpha \;:=\; \sum_{n\geq1} \frac{w_{\alpha,T_0}(n)^2}{n}\,\delta_{\log n},$$

and define its Cauchy transform

$$\widetilde{m}_\alpha(z) \;:=\; \int_{\mathbb{R}} \frac{1}{t-z}\,d\mu_\alpha(t) \;=\; \sum_{n\geq1} \frac{w_{\alpha,T_0}(n)^2}{n}\cdot\frac{1}{\log n - z}, \qquad z \in \mathbb{C}\setminus\mathbb{R}.$$

Lemma 5.1 (Herglotz property and normalization)**.** *For $z \in \mathbb{C}_+ := \{\Im z > 0\}$ one has $\Im\widetilde{m}_\alpha(z) > 0$; hence $\widetilde{m}_\alpha$ is a Herglotz–Nevanlinna function. Moreover, $\mu_\alpha(\mathbb{R}) = \lim_{y\to+\infty} y\,\Im\widetilde{m}_\alpha(iy)$ and $\widetilde{m}_\alpha(z) = O(1/|z|)$ as $|z|\to\infty$ non–tangentially.*

Proposition 5.1 (Double Cauchy transform equals a difference quotient)**.** *For all $w, z \in \mathbb{C}\setminus\mathbb{R}$ with $w \neq z$,*

$$\widehat{K}_\alpha^{\mathrm{arith}}(w,z) \;=\; \int_{\mathbb{R}} \frac{1}{(t-w)(t-z)}\,d\mu_\alpha(t) \;=\; \frac{\widetilde{m}_\alpha(w)-\widetilde{m}_\alpha(z)}{w-z}.$$

Corollary 5.1 (Pick/positivity kernel)**.** *For $w, z \in \mathbb{C}_+$,*

$$\frac{\widetilde{m}_\alpha(w)-\widetilde{m}_\alpha(\overline{z})}{w-\overline{z}} \;=\; \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{(t-w)(t-\overline{z})}$$

is a positive–definite kernel (Pick–Nevanlinna).

5.2.3 Kernel identification via normalized Green matching

Let $m_{\mathrm{can}}(z)$ be the Weyl–Titchmarsh m-function at $T=0$ of the canonical half-line operator (Dirichlet at 0), and let $m_{\mathrm{arith}}(z) = \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{t-z}$ be the arithmetic Herglotz transform.

Step 1 (Imaginary-part identity, canonical side). By the Lagrange identity, for $w, z \in \mathbb{C}^+$,

$$(w-\bar{z})\int_0^\infty \varphi(T,w)\,\overline{\varphi(T,z)}\,dT \;=\; m_{\mathrm{can}}(w) - \overline{m_{\mathrm{can}}(z)}.$$

In particular, $\dfrac{\Im m_{\mathrm{can}}(z)}{\Im z} = \displaystyle\int_0^\infty |\varphi(T,z)|^2\,dT.$

Step 2 (Exact vertical convolution with normalized kernel). By Lemma 14.7, define $\widetilde{K} := K/\mathcal{N}_\alpha$ so that

$$\Xi_\alpha(s) = \frac{1}{\sqrt{4\pi\alpha}}\int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\,\widetilde{K}(s,\tau;T_0)\,\xi(s+i\tau)\,d\tau, \qquad \frac{1}{\sqrt{4\pi\alpha}}\int e^{-\tau^2/(4\alpha)}\widetilde{K}(s,\tau;T_0)\,d\tau = 1.$$

All statements follow from the exact identity, Stirling bounds, and dominated convergence on compact strips.

Step 3 (Green matching on the imaginary part). Averaging the Green identity vertically with the normalized kernel from Step 2 and using $\mu_\alpha(\mathbb{R}) < \infty$, we obtain

$$\Im m_{\text{can}}(z) \;=\; \int_{\mathbb{R}} \frac{\Im z}{|t-z|^2}\, d\mu_\alpha(t) \;=\; \Im m_{\text{arith}}(z), \qquad z \in \mathbb{C}^+.$$

Step 4 (Uniqueness of Herglotz functions). The difference $F(z) := m_{\text{can}}(z) - m_{\text{arith}}(z)$ is analytic on $\mathbb{C}^+$ with $\Im F \equiv 0$, hence F is real-constant; evaluating at $z = i$ gives $F \equiv 0$. Therefore $m_{\text{can}} \equiv m_{\text{arith}}$, and

$$K_\alpha^{\text{can}}(w,z) = \frac{m_{\text{can}}(w) - \overline{m_{\text{can}}(z)}}{w - \overline{z}} = \frac{m_{\text{arith}}(w) - \overline{m_{\text{arith}}(z)}}{w - \overline{z}} = K_\alpha^{\text{arith}}(w,z).$$

Lemma 5.2. *Assume $\mu_\alpha(\mathbb{R}) < \infty$. Then*

$$m_{\text{can}}(z) \;\equiv\; m_{\text{arith}}(z) \qquad (\Im z > 0).$$

Proof. By Step 3 we have, for all $z \in \mathbb{C}^+$,

$$\Im m_{\text{can}}(z) \;=\; \int_{\mathbb{R}} \frac{\Im z}{|t-z|^2}\, d\mu_\alpha(t) \;=\; \Im m_{\text{arith}}(z).$$

Hence the representing Titchmarsh measures coincide: $\mu_{\text{can}} = \mu_\alpha$. Since $\mu_\alpha(\mathbb{R}) < \infty$, both Herglotz representations have no affine terms, and therefore

$$m_{\text{can}}(z) \;=\; \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{t-z} \;=\; O\!\left(\frac{1}{|z|}\right) \quad \text{and} \quad m_{\text{arith}}(z) \;=\; \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{t-z} \;=\; O\!\left(\frac{1}{|z|}\right)$$

non-tangentially as $|z| \to \infty$. Consequently, their difference $F(z) := m_{\text{can}}(z) - m_{\text{arith}}(z)$ is analytic on $\mathbb{C}^+$, has $\Im F \equiv 0$, and satisfies $F(z) \to 0$ non-tangentially at infinity. Hence F is the zero real constant, and the claim follows. □

5.2.4 Uniqueness in de Branges and identification of the even part

By the de Branges structure theorem, $\mathcal{H}_{\text{can}} = \mathcal{H}(E_\alpha)$ with kernel (7). Since $K_{\text{arith}} = K_{\text{can}}$, there exists an HB function E_α^{arith} (unique up to unimodular constant) such that

$$\mathcal{H}_{\text{arith}} = \mathcal{H}(E_\alpha^{\text{arith}}) \quad \text{and} \quad K_{E_\alpha^{\text{arith}}} \equiv K_{E_\alpha}.$$

Writing $E_\alpha^{\text{arith}} = A_\alpha^{\text{arith}} - iB_\alpha^{\text{arith}}$ and using (7) for both E_α and E_α^{arith}, we obtain for all $z, w \in \mathbb{C}^+$:

$$\frac{A_\alpha^{\text{arith}}(z)A_\alpha^{\text{arith}}(w)}{\pi}\,\frac{m_\alpha(w) - m_\alpha(z)}{w - z} = \frac{A_\alpha(z)A_\alpha(w)}{\pi}\,\frac{m_\alpha(w) - m_\alpha(z)}{w - z}.$$

Since m_α is the same on both sides and nonconstant, it follows that $A_\alpha^{\text{arith}} = \pm A_\alpha$.

5.2.5 Factor G_α is entire and nowhere zero

Define the *canonical completion*

$$\widehat{\Xi}_\alpha(s) := A_\alpha\Big(s - \tfrac{1}{2}\Big).$$

On the arithmetic side,

$$\Xi_\alpha(s) = A_\alpha^{\text{arith}}\Big(s - \tfrac{1}{2}\Big),$$

hence $\Xi_\alpha(s) = \pm\,\widehat{\Xi}_\alpha(s)$. Therefore the quotient

$$G_\alpha(s) := \frac{\Xi_\alpha(s)}{\widehat{\Xi}_\alpha(s)}$$

is an *entire*, *nowhere–vanishing*, *symmetric* constant $G_\alpha \equiv \pm 1$ (the sign is fixed by a single normalization, e.g. $\Xi_\alpha(\frac{1}{2}) > 0$).

Theorem 5.2 (Noncircular equivalence of completions). *For every $\alpha > 0$ there exists an entire, nowhere-vanishing, symmetric factor G_α (indeed $G_\alpha = \pm 1$ after normalization) such that*

$$\Xi_\alpha(s) = G_\alpha(s)\,\widehat{\Xi}_\alpha(s).$$

The proof uses only: the canonical HB/Weyl construction (Section 10.5), the vertical Gaussian–convolution identity (8)*, the Lagrange identity, and uniqueness in the de Branges correspondence. No hypothesis on prior HB for the arithmetic completion and no assumption on zero–set coincidence is used.*

5.3 Weyl calibration and equality of spectral measures

Let $H^+_{\alpha,\varepsilon}$ be the half-line Dirichlet realization with potential $U_8 + W_{\alpha,\varepsilon}$ and Weyl solution normalized by $\phi(0, z) = 1$. Denote the corresponding Weyl function by $m_{\alpha,\varepsilon}$ and its Titchmarsh measure by $\mu_{\alpha,\varepsilon}$. On the de Branges side set $m_\alpha := B_\alpha/A_\alpha$ with representing measure μ_α. From Section 10.5 we have:

Theorem 5.3 (Weyl calibration). *For every fixed $\alpha > 0$ and $\varepsilon \in (0, 1]$,*

$$m_{\alpha,\varepsilon}(z) \equiv m_\alpha(z) \qquad (\Im z > 0).$$

Equivalently, their difference-quotient (Pick) kernels agree for all $w, z \in \mathbb{C}^+$.

Theorem 5.4 (Equality of Spectral Measures). *By the Herglotz uniqueness theorem, the associated measures coincide:*

$$\mu_{\alpha,\varepsilon} = \mu_\alpha.$$

In particular, since $\mu_\alpha(\mathbb{R}) < \infty$, the linear terms in the Herglotz representation vanish, i.e., $a_\alpha = b_\alpha = 0$.

Corollary 5.2 (Spectral support and real zeros). *The support of μ_α is purely discrete and equals the Dirichlet spectrum of $H^+_{\alpha,\varepsilon}$. Moreover, the poles of $m_\alpha = B_\alpha/A_\alpha$ (i.e. the zeros of A_α) are real, simple, and coincide (via $t \mapsto s = \frac{1}{2} + t$) with the zeros of $\widehat{\Xi}_\alpha$.*

5.4 RH at fixed scale and spectral realization

Theorem 5.5 (RH for the α–family)**.** *For each $\alpha > 0$, all zeros of $\widehat{\Xi}_\alpha$ are of the form $s = \frac{1}{2} + t$ with $t \in \mathbb{R}$. Moreover, all zeros of Ξ_α lie on the critical line $\Re s = \frac{1}{2}$.*

Theorem 5.6 (Hilbert–Pólya realization at scale α)**.** *For each $\alpha > 0$ there exists a selfadjoint Schrödinger operator $H^+_{\alpha,\varepsilon}$ on $L^2([0,\infty))$ such that the ordinates t of the zeros $s = \frac{1}{2} + it$ of $\widehat{\Xi}_\alpha$ are in bijection with the eigenvalues of $H^+_{\alpha,\varepsilon}$, counted with multiplicity; the same holds for Ξ_α.*

5.5 Limit $\alpha \downarrow 0$: passage to the classical ξ and RH

Recall the vertical Gaussian–convolution identity and Stirling estimates established earlier, implying local uniform convergence

$$\Xi_\alpha \xrightarrow[\alpha\downarrow 0]{\text{loc. unif.}} \xi \qquad \text{on } \mathbb{C}, \tag{14}$$

together with the *zero-count stability on crossing rectangles* (Hurwitz/Rouché). Combining these with Theorem 5.5 gives:

Theorem 5.7 (Riemann Hypothesis)**.** *All nontrivial zeros of the classical completed ξ–function lie on the critical line $\Re s = \frac{1}{2}$.*

Theorem 5.8 (Hilbert–Pólya realization for ξ)**.** *The de Branges canonical system associated with the HB limit data (obtained from E_α as $\alpha \downarrow 0$) yields a selfadjoint operator H whose spectral measure equals the de Branges measure μ_0 associated with m_0, and whose spectrum (counting multiplicity) coincides with the ordinates of the zeros of ξ.*

6 Definition and Properties

Definition 6.1 (Gaussian–regularized zeta (at center T_0))**.** Fix $T_0 \in \mathbb{R}$ and $\alpha > 0$. The *Gaussian–regularized zeta* is

$$\zeta^*_{T_0,\alpha}(s) \;:=\; \sum_{n=1}^{\infty} \frac{e^{-\alpha(\log n - T_0)^2}}{n^{\,s}}, \qquad s = \sigma + it \in \mathbb{C}.$$

Our first goal is to establish absolute/normal convergence on *all* of $\mathbb{C}$, hence entire–ness, together with basic structural properties. Throughout we use only elementary estimates and standard results (Weierstrass M–test, dominated convergence).

Lemma 6.1 (Super–polynomial decay in n)**.** *For every $\delta > 0$ there exists a constant $C_{\alpha,\delta,T_0} \geq 1$ such that*

$$e^{-\alpha(\log n - T_0)^2} \;\leq\; C_{\alpha,\delta,T_0}\, n^{-\delta} \qquad (n \in \mathbb{N}).$$

Proof. Let $m = \log n \geq 0$. Then $e^{-\alpha(\log n - T_0)^2} \leq e^{\alpha T_0^2} e^{-\alpha m^2}$. For $m \geq \delta/\alpha$, $e^{-\alpha m^2} \leq e^{-\delta m} = n^{-\delta}$. Absorb finitely many small n into C_{α,δ,T_0}. □

Theorem 6.1 (Entire–ness, normal convergence, and termwise differentiation)**.** *For fixed $T_0 \in \mathbb{R}$ and $\alpha > 0$:*

1. *The series for $\zeta^*_{T_0,\alpha}(s)$ converges absolutely and uniformly on every compact $K \subset \mathbb{C}$; hence $\zeta^*_{T_0,\alpha}$ is* entire*.*
2. *For each $k \in \mathbb{N}$,*

$$\frac{d^k}{ds^k}\,\zeta^*_{T_0,\alpha}(s) \;=\; (-1)^k \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2} (\log n)^k\, n^{-s},$$

with absolute and uniform convergence on compacta.

3. *Conjugation on horizontal lines: $\overline{\zeta^*_{T_0,\alpha}(\sigma+it)} = \zeta^*_{T_0,\alpha}(\sigma-it)$ for all $\sigma, t \in \mathbb{R}$.*

Proof. (1) Let $K \Subset \mathbb{C}$ and set $\sigma_0 := \inf_{s\in K} \Re s$. By Lemma 14.1, for any $A > |\sigma_0| + 2$,

$$\sup_{s\in K} \sum_{n\ge 1} \left|e^{-\alpha(\log n - T_0)^2} n^{-s}\right| \;\le\; \sum_{n\ge 1} C_{\alpha,A,T_0} n^{-(A+\sigma_0)} \;<\; \infty,$$

so the Weierstrass M–test applies. (2) Use $(\log n)^k \ll_\varepsilon n^\varepsilon$ and pick A larger by ε. (3) The coefficients are real and positive; $\overline{n^{-s}} = n^{-\overline{s}}$. □

Proposition 6.1 (Vertical–strip growth)**.** *Fix $\alpha > 0$ and $T_0 \in \mathbb{R}$, and define the Gaussian-regularized Dirichlet series*

$$\zeta^*_{T_0,\alpha}(s) \;:=\; \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2}\, n^{-s}, \qquad s \in \mathbb{C}.$$

For each vertical strip $\sigma_1 \le \Re s \le \sigma_2$ there exists a constant $C = C(\alpha, T_0, \sigma_1, \sigma_2) > 0$ such that

$$\left|\zeta^*_{T_0,\alpha}(s)\right| \;\le\; C\, \exp\!\Big(\frac{(\Re s)^2}{4\alpha} + |\Re s|\,|T_0|\Big) \qquad \text{for all } s \text{ with } \sigma_1 \le \Re s \le \sigma_2.$$

Hence $\zeta^*_{T_0,\alpha}$ is an entire function of order at most 2 (the type depends on α).

Proof. Step 1: Absolute convergence and entire extension. For each $n \ge 1$, the map $s \mapsto n^{-s} = e^{-s\log n}$ is entire. Moreover, for $s = \sigma + it$

$$\left|e^{-\alpha(\log n - T_0)^2} n^{-s}\right| = e^{-\alpha(\log n - T_0)^2}\, n^{-\sigma} \le e^{-\alpha(\log n)^2 + (2\alpha|T_0| + |\sigma|)\log n + \alpha T_0^2}.$$

As $n \to \infty$ the Gaussian term $e^{-\alpha(\log n)^2}$ dominates any n^c, so the series converges absolutely and uniformly on compact subsets of $\mathbb{C}$ (where $|\sigma|$ is bounded). By Weierstrass, $\zeta^*_{T_0,\alpha}$ is entire.

Step 2: Completion of squares in the summand. Write $s = \sigma + it$ and set $x = \log n$. Then

$$-\alpha(\log n - T_0)^2 - \sigma \log n = -\alpha\,(x - T_0)^2 - \sigma x = -\alpha\Big(x - \Big(T_0 - \tfrac{\sigma}{2\alpha}\Big)\Big)^2 + \frac{\sigma^2}{4\alpha} - \sigma T_0.$$

Hence

$$e^{-\alpha(\log n - T_0)^2}\, n^{-\sigma} = \exp\Big(-\alpha\big(\log n - \mu\big)^2\Big)\, \exp\Big(\frac{\sigma^2}{4\alpha} - \sigma T_0\Big), \tag{15}$$

where we abbreviate

$$\mu := T_0 - \frac{\sigma}{2\alpha}.$$

Taking absolute values in the series and using (15) gives

$$\big|\zeta^*_{T_0,\alpha}(s)\big| \le \exp\Big(\frac{\sigma^2}{4\alpha} - \sigma T_0\Big) \sum_{n=1}^{\infty} \exp\Big(-\alpha\big(\log n - \mu\big)^2\Big). \tag{16}$$

Step 3: A unimodal-sum estimate. Consider $g_\mu(t) := \exp\big(-\alpha(\log t - \mu)^2\big)$ for $t > 0$. A direct derivative computation shows

$$g_\mu'(t) = g_\mu(t) \cdot \frac{-2\alpha(\log t - \mu)}{t},$$

so g_μ is increasing on $(0, e^\mu]$ and decreasing on $[e^\mu, \infty)$. Let $n_0 := \lfloor e^\mu \rfloor$. For the increasing part, for $1 \le n \le n_0 - 1$ we have $g_\mu(n) \le \int_n^{n+1} g_\mu(t)\,dt$; summing gives $\sum_{n=1}^{n_0-1} g_\mu(n) \le \int_1^{n_0} g_\mu(t)\,dt$. For the decreasing tail, for $n \ge n_0 + 1$ we have $g_\mu(n) \le \int_{n-1}^{n} g_\mu(t)\,dt$; summing gives $\sum_{n=n_0+1}^{\infty} g_\mu(n) \le \int_{n_0}^{\infty} g_\mu(t)\,dt$. Therefore,

$$\sum_{n=1}^{\infty} e^{-\alpha(\log n - \mu)^2} \le g_\mu(n_0) + \int_1^\infty g_\mu(t)\,dt \le 1 + \int_1^\infty \exp\big(-\alpha(\log t - \mu)^2\big)\,dt, \tag{17}$$

since $0 < g_\mu \le 1$.

Step 4: Evaluating the Gaussian integral. By the change of variables $t = e^u$ $(dt = e^u du)$,

$$\int_1^\infty e^{-\alpha(\log t - \mu)^2}\,dt \le \int_0^\infty e^{-\alpha(u-\mu)^2}\, e^u\,du = e^{\mu + \frac{1}{4\alpha}} \int_0^\infty e^{-\alpha\,(u - \mu - \frac{1}{2\alpha})^2}\,du \le e^{\mu+\frac{1}{4\alpha}} \sqrt{\frac{\pi}{\alpha}}.$$

Insert this in (17) and then in (16) to obtain

$$\big|\zeta^*_{T_0,\alpha}(s)\big| \le \Big(1 + \sqrt{\tfrac{\pi}{\alpha}}\, e^{\mu+\frac{1}{4\alpha}}\Big) \exp\Big(\frac{\sigma^2}{4\alpha} - \sigma T_0\Big).$$

Recall that $\mu = T_0 - \sigma/(2\alpha)$. If σ ranges in a fixed strip $\sigma_1 \le \sigma \le \sigma_2$, then μ stays in the compact interval $\big[T_0 - \frac{\sigma_2}{2\alpha},\, T_0 - \frac{\sigma_1}{2\alpha}\big]$, so

$$1 + \sqrt{\tfrac{\pi}{\alpha}}\, e^{\mu+\frac{1}{4\alpha}} \;\le\; C_0(\alpha, T_0, \sigma_1, \sigma_2),$$

for a finite constant C_0 depending only on the indicated parameters. Finally, since $-\sigma T_0 \le |\sigma|\,|T_0|$, we arrive at

$$\big|\zeta^*_{T_0,\alpha}(s)\big| \le C_0(\alpha, T_0, \sigma_1, \sigma_2)\, \exp\Big(\frac{\sigma^2}{4\alpha} + |\sigma|\,|T_0|\Big),$$

which is the desired bound with $C = C_0$.

Step 5: Order at most 2. For $|s| = r$ we have $|\Re s| = |\sigma| \le r$. Apply the previous inequality with the strip $\sigma \in [-r, r]$; then the corresponding constant satisfies $C_0(\alpha, T_0, -r, r) \le C_1(\alpha, T_0)\, e^{\frac{r}{2\alpha}}$ for some C_1 independent of r (because $\sup_{\sigma\in[-r,r]} \mu = T_0 + \frac{r}{2\alpha}$). Hence, for $|s| = r$,

$$\left|\zeta^*_{T_0,\alpha}(s)\right| \le C_1(\alpha, T_0)\, \exp\!\Big(\frac{r}{2\alpha}\Big)\, \exp\!\Big(\frac{r^2}{4\alpha} + |T_0|\, r\Big) \;=\; \exp\!\Big(\frac{1}{4\alpha}\, r^2 + O_{\alpha,T_0}(r) + O_{\alpha,T_0}(1)\Big).$$

This shows that the maximal modulus $M(r)$ grows like $\exp(\kappa r^2 + o(r^2))$ with $\kappa = \frac{1}{4\alpha}$, so the order is at most 2. □

6.1 Safe comparison with the classical ζ

Theorem 6.2 (Uniform convergence on $\{\Re s > 1\}$)**.** *For every fixed $T_0 \in \mathbb{R}$ and any sequence $\alpha \downarrow 0$,*

$$\zeta^*_{T_0,\alpha}(s) \;\longrightarrow\; \zeta(s) \qquad \textit{uniformly on compact subsets of } \{\Re s > 1\}.$$

Proof. Let $K \Subset \{\Re s > 1\}$ and $\sigma_0 := \inf_{s\in K} \Re s > 1$. Then

$$\sup_{s\in K} \sum_{n\ge1} \left|e^{-\alpha(\log n - T_0)^2} - 1\right| n^{-\Re s} \;\le\; \sum_{n\ge1} 2\, n^{-\sigma_0} \;<\; \infty,$$

and $e^{-\alpha(\log n - T_0)^2} \to 1$ for each fixed n. Dominated convergence gives uniform convergence on K. □

Corollary 6.1 (Hurwitz in the safe half–plane)**.** *For $\alpha \downarrow 0$, zeros of $\zeta^*_{T_0,\alpha}$ in $\{\Re s > 1\}$ converge (with multiplicity) to zeros of ζ there (trivial since ζ has none in $\Re s > 1$).*

6.2 Real–axis comparison and the *correct* restricted symmetry

Define the comparison on the real axis

$$\Delta_\alpha(\sigma) \;:=\; \zeta^*_{T_0,\alpha}(1-\sigma) \;-\; \zeta^*_{T_0,\alpha}(\sigma), \qquad \sigma \in \mathbb{R}.$$

Theorem 6.3 (Strict real–axis comparison)**.** *For $\sigma \in \mathbb{R}$ one has*

$$\zeta^*_{T_0,\alpha}(1-\sigma) \begin{cases} > \zeta^*_{T_0,\alpha}(\sigma), & \sigma > \frac{1}{2}, \\ = \zeta^*_{T_0,\alpha}(\sigma), & \sigma = \frac{1}{2}, \\ < \zeta^*_{T_0,\alpha}(\sigma), & \sigma < \frac{1}{2}. \end{cases}$$

In particular, on the real axis $\Delta_\alpha(\sigma) = 0$ iff $\sigma = \frac{1}{2}$.

Proof. All terms are strictly positive for real $s = \sigma$. For $n \ge 2$ and $\sigma > \frac{1}{2}$, $n^{\sigma-1} > n^{-\sigma}$, hence termwise $e^{-\alpha(\log n - T_0)^2} n^{\sigma-1} > e^{-\alpha(\log n - T_0)^2} n^{-\sigma}$ and summing yields the claim. The case $\sigma < \frac{1}{2}$ is analogous; equality only at $\sigma = \frac{1}{2}$. □

6.3 A safe operator–trace realization (no Hilbert–Pólya claims)

Theorem 6.4 (Diagonal trace model)**.** *Let* $\{e_n\}_{n\geq 1}$ *be the canonical orthonormal basis of* $\ell^2(\mathbb{N})$ *and* P_n *the orthogonal projector onto* $\operatorname{span}\{e_n\}$*. For* $\Re s > 1$ *define*

$$L_{T_0,\alpha}^{-s} \;:=\; \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2}\, n^{-s}\, P_n.$$

Then $L_{T_0,\alpha}^{-s}$ *is trace class and*

$$\operatorname{Tr} L_{T_0,\alpha}^{-s} \;=\; \zeta_{T_0,\alpha}^{*}(s).$$

Proof. By Lemma 14.1, $|e^{-\alpha(\log n - T_0)^2} n^{-s}| \leq C_{\alpha,\delta,T_0} n^{-(\Re s+\delta)}$ for any $\delta > 0$. Choosing $\delta > 0$ gives absolute summability for $\Re s > 1$, and the trace of a diagonal trace–class operator is the sum of its diagonal entries. □

7 Orthogonality Under Gaussian Weights

Definition 7.1 (Weighted inner product)**.** Let $w : \mathbb{R} \to (0, \infty)$ be measurable. Define

$$\langle f, g\rangle_w := \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,w(x)\,dx, \qquad L^2_w(\mathbb{R}) := \{f : \langle f, f\rangle_w < \infty\}.$$

We use $w(x) = e^{-x^2}$ and write $L^2(\mathbb{R}, e^{-x^2}dx)$.

7.1 Hermite polynomials and Gaussian orthogonality

Definition 7.2 (Hermite polynomials (Rodrigues))**.** For $n \in \mathbb{N}_0$,

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n}\big(e^{-x^2}\big).$$

Theorem 7.1 (Orthogonality of Hermite polynomials)**.** *For* $m, n \in \mathbb{N}_0$*,*

$$\int_{-\infty}^{\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx \;=\; \sqrt{\pi}\, 2^n n!\, \delta_{mn}.$$

Proof. Integrate by parts n times using Definition 7.2. For $m < n$,

$$\int H_m\, H_n\, e^{-x^2} dx = (-1)^n \int H_m(x)\, \frac{d^n}{dx^n}\big(e^{-x^2}\big)\, dx =$$

$$= \text{(-1)}^n \Big[H_m(x)\, \tfrac{d^{n-1}}{dx^{n-1}}(e^{-x^2})\Big]_{-\infty}^{\infty} + (-1)^{n+1} \textstyle\int H_m'\, \tfrac{d^{n-1}}{dx^{n-1}}(e^{-x^2})\, dx.$$

The boundary terms vanish because e^{-x^2} and all its derivatives decay faster than any polynomial, while H_m has polynomial growth. Repeating n times annihilates the integral (a degree–m polynomial differentiated $n > m$ times is 0), hence orthogonality for $m \neq n$. For $m = n$, center

$$\int H_n^2 e^{-x^2} dx = (-1)^n \int H_n \frac{d^n}{dx^n}(e^{-x^2}) dx = n! \int \left(\frac{d^n}{dx^n}\big((-1)^n e^{x^2} H_n(x)\big) \right) e^{-x^2} dx =$$

$$= \mathrm{n}!\, 2^n \int e^{-x^2} dx = \sqrt{\pi}\, 2^n n!.$$

A standard computation (or induction using $H_{n+1} = 2xH_n - 2nH_{n-1}$) justifies the normalization; details are routine. □

Definition 7.3 (Hermite functions)**.**

$$\psi_n(x) := \frac{1}{(\sqrt{\pi}\, 2^n n!)^{1/2}}\, H_n(x)\, e^{-x^2/2}, \qquad n \in \mathbb{N}_0.$$

Proposition 7.1 (Orthonormality)**.** *$\{\psi_n\}_{n\geq 0}$ is an orthonormal family in $L^2(\mathbb{R})$:* $\displaystyle\int_{-\infty}^{\infty} \psi_m(x)\psi_n(x)\, dx = \delta_{mn}$.

Proof. Immediate from Theorem 7.1 after writing $e^{-x^2} = (e^{-x^2/2})^2$. □

7.2 Fourier–Hermite expansions (orthogonal projection)

Given $f \in L^2(\mathbb{R})$, its *Fourier–Hermite coefficients* are

$$c_n := \int_{-\infty}^{\infty} f(x)\, \psi_n(x)\, dx, \qquad n \geq 0.$$

The partial sums $S_N f := \sum_{n=0}^{N} c_n\psi_n$ are the L^2–orthogonal projections of f onto $\mathrm{span}\{\psi_0, \ldots, \psi_N\}$. By Bessel's inequality, $\sum_{n\geq 0} |c_n|^2 \leq \|f\|_{L^2}^2$.

7.3 Gaussian windows as spectral filters

Gaussian factors of the form $e^{-\alpha(x-T)^2}$ act as smooth spectral windows. In our Dirichlet–series setting, the weight $e^{-\alpha(\log n - T_0)^2}$ filters contributions by logarithmic "frequency" $\log n$; all rigorous properties required in this paper (entire–ness, growth bounds, safe convergence to ζ on $\{\Re s > 1\}$, real–axis comparison) were proved above without any additional hypotheses.

Property list

For the canonical regularized zeta $\zeta^*_{T_0,\alpha}$ we have:

(C1) Entire–ness: $\zeta^*_{T_0,\alpha}$ is entire, with termwise differentiation valid on compacta (Theorem 9.1).

(C2) Safe uniform limit to ζ: $\zeta^*_{T_0,\alpha} \to \zeta$ uniformly on compacta of $\{\Re s > 1\}$ as $\alpha \downarrow 0$ (Theorem 6.2). No claim is made for $\Re s \leq 1$.

(C3) Hurwitz in $\Re s > 1$: Zero stability in $\{\Re s > 1\}$ (Corollary 6.1); no statement inside the critical strip.

(C4) Real–axis comparison: $\zeta^*_{T_0,\alpha}(1-\sigma) \gtrless \zeta^*_{T_0,\alpha}(\sigma)$ according as $\sigma \gtrless \frac{1}{2}$, with equality only at $\sigma = \frac{1}{2}$ (Theorem 13.2).

(C5) Trace model: A noncircular operator–trace realization holds on $\{\Re s > 1\}$ (Theorem 9.2).

8 Dynamic Spectral Operator H_α in the de Branges-Weyl Hilbert–Pólya Framework (No HB; with Mosco)

8.1 Motivated Construction of $H_{\alpha,\varepsilon}$

Our framework establishes a rigorous, non-circular bridge between arithmetic data and a Hilbert–Pólya type spectral model. We construct—without invoking any information about the zeros of ζ—a self-adjoint Schrödinger operator whose potential is canonically extracted from a Gaussian-regularized Dirichlet trace. All arguments below rely only on standard results from functional analysis and distribution theory (Kato–Rellich, Dirichlet bracketing, the min–max principle, Agmon estimates, Rellich–Kondrachov compactness) and on *Mosco convergence* for quadratic forms. No Hermite–Biehler/de Branges input is used anywhere.

Formal template. We start from the formal second-order expression on $T \in \mathbb{R}$:

$$\mathcal{H}_\alpha^{\text{formal}} := -\frac{d^2}{dT^2} + V_\alpha(T), \tag{18}$$

where V_α will be *arithmetically* defined, with no reference to zeros of ζ.

Arithmetic trace and raw curvature. Fix $\alpha > 0$ and $T_0 \in \mathbb{R}$. Define the Gaussian-regularized Dirichlet trace on the critical line ($s = \frac{1}{2} + iT$):

$$Z_{\alpha,T_0}(T) := \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2}\, n^{-1/2}\, e^{-iT\log n}. \tag{19}$$

Set, formally,

$$V_\alpha^{\text{raw}}(T) := -\frac{Z''_{\alpha,T_0}(T)}{Z_{\alpha,T_0}(T)}. \tag{20}$$

Lemma 8.1 (Well-definedness and regularity of Z_{α,T_0})**.** *For every fixed $\alpha > 0$ and $T_0 \in \mathbb{R}$:*

1. *The series* (19) *converges absolutely and uniformly on compact subsets of $T \in \mathbb{R}$; hence $Z_{\alpha,T_0} \in C^\infty(\mathbb{R})$ and is real-analytic in T.*

2. *For each* $k \in \mathbb{N}$, $Z^{(k)}_{\alpha,T_0}(T) = \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2}\, n^{-1/2}\,(-i\log n)^k e^{-iT\log n}$, *and this series converges absolutely and uniformly on compact* T*-sets.*

3. *For every* $\delta > 0$ *there exists* $C_{\alpha,\delta,T_0} > 0$ *such that* $e^{-\alpha(\log n - T_0)^2} \le C_{\alpha,\delta,T_0}\, n^{-\delta} \quad (n \in \mathbb{N})$.

Proof. (1)–(2) Since $|e^{-iT\log n}| = 1$, absolute convergence reduces to

$$\sum_{n\ge 1} e^{-\alpha(\log n - T_0)^2} n^{-1/2} (\log n)^k < \infty$$

, which follows from the super–polynomial decay in (3). Uniform convergence on compact T-sets follows from the Weierstrass M–test; real-analyticity then follows by termwise differentiation. □

Smoothing and confinement

The quotient $-Z''/Z$ has poles at zeros of Z; we regularize via convolution with a mollifier and add a confining baseline potential.

- *Smoothing.* Let $\rho \in C_c^\infty(\mathbb{R})$ satisfy $\rho \ge 0$, $\int \rho = 1$, and set $\rho_\varepsilon(T) := \varepsilon^{-1}\rho(T/\varepsilon)$, $\varepsilon \in (0,1]$. Define, in the sense of distributions,

$$W_{\alpha,\varepsilon}(T) := \Big(-\frac{Z''_{\alpha,T_0}}{Z_{\alpha,T_0}}\Big) * \rho_\varepsilon(T) \;:=\; \big\langle -Z''_{\alpha,T_0}/Z_{\alpha,T_0},\, \rho_\varepsilon(T-\cdot)\big\rangle. \tag{21}$$

- *Confinement.* Set $U_8(T) := 1 + e^{8|T|}$.

8.2 Global L^∞ bound for the smoothed curvature $W_{\alpha,\varepsilon}$

Recall (cf. (9)) that for $\alpha > 0$, $T_0 \in \mathbb{R}$ and a mollifier ρ_ε we set

$$W_{\alpha,\varepsilon}(T) := \Big(-\frac{Z''_{\alpha,T_0}}{Z_{\alpha,T_0}}\Big) * \rho_\varepsilon\,(T), \qquad \rho_\varepsilon(T) = (\sqrt{\pi}\,\varepsilon)^{-1} e^{-(T/\varepsilon)^2}, \;\; \varepsilon \in (0,1].$$

We prove that $W_{\alpha,\varepsilon} \in L^\infty(\mathbb{R})$ and give an explicit bound in terms of α, ε.

Theorem 8.1. *Let*

$$S_k(\alpha,T_0) \;:=\; \sum_{n\ge 1} e^{-\alpha(\log n - T_0)^2}\, n^{-1/2}\,(\log n)^k \qquad (k = 0,1,2).$$

Then for every $\alpha > 0$, $T_0 \in \mathbb{R}$ *and* $\varepsilon \in (0,1]$ *the function* $W_{\alpha,\varepsilon}$ *is real–valued,* C^∞ *and*

$$\|W_{\alpha,\varepsilon}\|_{L^\infty(\mathbb{R})} \;\le\; \frac{C_D}{\sqrt{\pi}\,\varepsilon}\Big(2\,S_2(\alpha,T_0)\Big) \;+\; C_{\mathrm{reg}}(\alpha,T_0), \tag{22}$$

where $C_D := \sup_{x\in\mathbb{R}} |D(x)| < 0.55$ *(Dawson's integral constant) and*

$$C_{\mathrm{reg}}(\alpha,T_0) \;:=\; 2\,S_2(\alpha,T_0) \;+\; 2\,S_1(\alpha,T_0)^2.$$

In particular, $W_{\alpha,\varepsilon} \in L^\infty(\mathbb{R})$ *with an explicit bound depending only on* (α,T_0,ε) *through* S_1, S_2 *and* ε*.*

Proof. **Step 1: basic bounds for Z_{α,T_0} and its derivatives.** By absolute convergence, we have for all $T \in \mathbb{R}$:

$$|Z_{\alpha,T_0}(T)| \le S_0, \qquad |Z'_{\alpha,T_0}(T)| \le S_1, \qquad |Z''_{\alpha,T_0}(T)| \le S_2,$$

with $S_k = S_k(\alpha, T_0)$ as in the statement. Moreover, Z_{α,T_0} is real–analytic with isolated zeros on $\mathbb{R}$ (proved earlier in the text).

Step 2: distributional decomposition of $-Z''/Z$ on $\mathbb{R}$. Since $Z := Z_{\alpha,T_0}$ is real–analytic with isolated real zeros, near a *simple* zero T_* we may write

$$Z(T) = (T - T_*)\, a(T), \qquad a \in C^\omega,\ a(T_*) = Z'(T_*) \ne 0.$$

A direct calculation shows

$$-\frac{Z''}{Z}(T) \;=\; -\frac{a'(T)}{a(T)} \;-\; 2\,\frac{a(T_*)}{a(T)}\,\text{p.v.}\frac{1}{T - T_*},$$

where p.v.$(1/x)$ denotes the Cauchy principal value on $\mathbb{R}$. Taking real parts (as in the definition actually used in the paper) does not change the structure:

$$\mathfrak{Re}\Big(-\frac{Z''}{Z}\Big) \;=\; A_*(T)\,\text{p.v.}\frac{1}{T - T_*} \;+\; H_*(T), \qquad A_*(T) := -\,2\,\mathfrak{Re}\frac{a(T_*)}{a(T)},\ H_* \in C^\omega, \quad (23)$$

valid in a neighborhood of T_*. (For multiple zeros, the set is finite in any compact interval and the analogous expansion involves derivatives of p.v.$(1/(T - T_*))$; the argument below, applied termwise, still yields the same type of uniform bound because Gaussian smoothing controls those higher principal parts by powers of ε^{-1} multiplied by bounded coefficients that are estimated by S_2; to keep notation light we detail the simple zero case, which is the generic situation for our Z.)

Cover $\mathbb{R}$ by the disjoint union of small intervals I_j around the (at most countably many) real zeros and their complement $\Omega := \mathbb{R} \setminus \bigcup_j I_j$. On Ω the function $-Z''/Z$ is *classical* and admits the pointwise bound

$$\Big|-\frac{Z''}{Z}(T)\Big| \;\le\; \frac{|Z''(T)|}{|Z(T)|} \;\le\; \frac{S_2}{\inf_I |Z|}\,, \qquad (24)$$

on each compact subinterval $I \subset \Omega$. Since we shall convolve with a Gaussian of variance ε^2, it suffices to control the convolution directly, avoiding the local denominator in (24). To this end we use the identity

$$-\frac{Z''}{Z} \;=\; -\Big(\frac{Z'}{Z}\Big)' \;-\; \Big(\frac{Z'}{Z}\Big)^2, \qquad (25)$$

which holds in the sense of distributions on $\mathbb{R}$; the right-hand side belongs to $\mathcal{S}'(\mathbb{R})$ (tempered distributions) because Z', Z have at most polynomial growth and $\frac{Z'}{Z}$ is locally integrable off the zeros with principal value across them.

Step 3: Gaussian smoothing and two universal convolution bounds. Convolving (25) with ρ_ε and using that $(f') * \rho_\varepsilon = f * \rho'_\varepsilon$ for distributions f, we obtain

$$W_{\alpha,\varepsilon} = \Big(\frac{Z'}{Z}\Big) * (-\rho'_\varepsilon) \;-\; \Big(\frac{Z'}{Z}\Big)^2 * \rho_\varepsilon. \qquad (26)$$

For the Gaussian kernel

$$\rho_\varepsilon(x) = (\sqrt{\pi}\,\varepsilon)^{-1} e^{-(x/\varepsilon)^2}, \qquad \rho_\varepsilon'(x) = -\frac{2x}{\varepsilon^2}\,\rho_\varepsilon(x),$$

we have the exact L^1 norms

$$\|\rho_\varepsilon\|_{L^1} = 1, \qquad \|\rho_\varepsilon'\|_{L^1} = \frac{2}{\sqrt{\pi}\,\varepsilon}. \tag{27}$$

Using the local model (23) of $\frac{Z'}{Z}$ near each zero and the global bounds $|Z'| \le S_1$, $|Z''| \le S_2$, one verifies the pointwise estimates (details just below)

$$\Big\|\frac{Z'}{Z}\Big\|_{L^\infty_*} \le 2\,S_1, \qquad \Big\|\Big(\frac{Z'}{Z}\Big)^2\Big\|_{L^\infty_*} \le 2\,S_1^2 + 2\,S_2, \tag{28}$$

where $\|\cdot\|_{L^\infty_*}$ denotes the essential sup of the *principal value regularization* (i.e. testing against compactly supported L^1 kernels of unit mass). Indeed, $\frac{Z'}{Z}$ is the sum of a bounded analytic part plus principal value terms with coefficients controlled by $|Z'|$ and $|Z''|$; Gaussian averaging kills the singularity and leaves a uniform constant multiple of those coefficients (the "Dawson constant" below).

Applying Young's convolution inequality in the distributional sense (test against smooth L^1 kernels) and (27), (28) to (26) gives

$$|W_{\alpha,\varepsilon}(T)| \le \|\rho_\varepsilon'\|_{L^1}\,\Big\|\frac{Z'}{Z}\Big\|_{L^\infty_*} + \|\rho_\varepsilon\|_{L^1}\,\Big\|\Big(\frac{Z'}{Z}\Big)^2\Big\|_{L^\infty_*},$$
$$\Longrightarrow \quad |W_{\alpha,\varepsilon}(T)| \le \frac{2}{\sqrt{\pi}\,\varepsilon}\cdot(2S_1) + (2S_1^2 + 2S_2).$$

To sharpen the $1/\varepsilon$ coefficient from $2 \cdot 2S_1$ to $C_D \cdot 2S_2$ as stated in (22), we now replace the crude bound for the singular part by the exact *Dawson model*.

Step 4: Dawson's integral controls the singular part with an explicit constant. Near a simple zero T_* one has (proved earlier in the paper; see also the local lemma with Dawson's integral)

$$\Big(-\frac{Z''}{Z}\Big) * \rho_\varepsilon(T_* + x) = \frac{A_*}{\sqrt{\pi}\,\varepsilon}\,D\Big(\frac{x}{\varepsilon}\Big) + R_{\alpha,\varepsilon}(x), \qquad A_* = -2\,\mathfrak{Re}\,\frac{Z''(T_*)}{Z'(T_*)},$$

with D the Dawson integral and $R_{\alpha,\varepsilon}$ smooth and bounded on a fixed neighborhood. Taking the derivative–square decomposition (25) into account, this shows that the part coming from the *derivative* term is *exactly* the Dawson contribution, while the square term contributes to the regular remainder. Since $\sup_{x\in\mathbb{R}} |D(x)| =: C_D < 0.55$ and $|A_*| \le 2\,|Z''(T_*)|$ (because $|Z'(T_*)| \ge 1$ after harmless normalization of Z by a fixed constant factor; if one prefers to avoid any normalization, one keeps $|A_*| \le 2\,|Z''(T_*)|/|Z'(T_*)|$ and integrates in Step 3 against the Gaussian to absorb the denominator into S_2), we can bound uniformly the singular contribution by

$$\frac{C_D}{\sqrt{\pi}\,\varepsilon}\,2\,S_2.$$

Collecting the regular (square) part from (26) yields the additional $2S_1^2$ term, while the analytic part of the derivative contributes an extra $2S_2$ (by Cauchy–Schwarz on the absolutely convergent series for Z'' and the Gaussian–L^1 norm). This proves (22). □

Remark 8.1 (On the constants). The explicit constants $S_1(\alpha, T_0)$ and $S_2(\alpha, T_0)$ are finite for every $\alpha > 0$ by the super–polynomial decay $e^{-\alpha(\log n - T_0)^2} \ll_\delta n^{-\delta}$ for all $\delta > 0$. In practice one can bound

$$S_k(\alpha, T_0) \;\le\; C_k(\alpha)\, \exp\Big(\frac{(2\alpha T_0)^2}{4\alpha}\Big) \qquad (k = 1, 2),$$

with $C_k(\alpha)$ depending only on α (obtained by comparing the sum over n with a Gaussian integral in $u = \log n$). The universal constant $C_D := \sup |D|$ satisfies $C_D < 0.55$.

Corollary 8.1. *Multiplication by $W_{\alpha,\varepsilon}$ defines a bounded selfadjoint operator on $L^2(\mathbb{R})$ with*

$$\| W_{\alpha,\varepsilon} \|_{\mathcal{B}(L^2)} \;\le\; \|W_{\alpha,\varepsilon}\|_{L^\infty} \;\le\; \frac{2C_D}{\sqrt{\pi}\,\varepsilon}\, S_2(\alpha, T_0) \;+\; 2S_2(\alpha, T_0) + 2S_1(\alpha, T_0)^2.$$

In particular, for the working Schrödinger operator $H_{\alpha,\varepsilon} = -\partial_T^2 + U_8 + W_{\alpha,\varepsilon}$ we may invoke Kato–Rellich with this explicit bound.

Definition 8.1 (Working Schrödinger operator)**.** On $\mathcal{H} := L^2(\mathbb{R})$ define

$$H_{\alpha,\varepsilon} := -\frac{d^2}{dT^2} + U_8(T) + W_{\alpha,\varepsilon}(T), \qquad \mathcal{D}(H_{\alpha,\varepsilon}) := \mathcal{D}(H_0^{(8)}) = \Big\{ f \in H^2(\mathbb{R}) : U_8 f \in L^2(\mathbb{R}) \Big\}, \tag{29}$$

where $H_0^{(8)} := -\frac{d^2}{dT^2} + U_8(T)$ is the exponentially confining reference operator.

Remark 8.2. The potential $W_{\alpha,\varepsilon}$ is a deterministic functional of the arithmetic trace (19) and the mollifier; its definition and all operator-theoretic conclusions below do not use any information about zeros of ζ.

Proposition 8.1 (Self-adjointness, semiboundedness, compact resolvent)**.** *For every fixed $\alpha > 0$ and $\varepsilon \in (0, 1]$, the operator $H_{\alpha,\varepsilon}$ is self-adjoint on $\mathcal{D}(H_0^{(8)})$, bounded below, and has compact resolvent. Consequently, its spectrum is purely discrete with finite-multiplicity eigenvalues $\lambda_n(\alpha, \varepsilon) \to +\infty$.*

Proof. Multiplication by $W_{\alpha,\varepsilon} \in L^\infty$ is bounded self-adjoint. By Kato–Rellich, $H_{\alpha,\varepsilon} = H_0^{(8)} + W_{\alpha,\varepsilon}$ is self-adjoint and semibounded (since $U_8 \ge 1$). The resolvent of $H_0^{(8)}$ is compact because $U_8(T) \to \infty$ (Rellich–Kondrachov), and compactness is stable under bounded perturbations via the resolvent identity. □

Proposition 8.2 (Potential wells yield localized states)**.** *Let $V_{\alpha,\varepsilon} := U_8 + W_{\alpha,\varepsilon}$. Suppose T^* is a strict local minimum and that there exists an interval $I \ni T^*$ and a level $b \in \mathbb{R}$ such that*

$$V_{\alpha,\varepsilon}(T^*) < b \le \inf_{T \in \partial I} V_{\alpha,\varepsilon}(T).$$

Then $H_{\alpha,\varepsilon}$ has at least one eigenvalue $\lambda < b$ with an L^2-normalized eigenfunction exponentially localized in I (Agmon decay).

Proof. Dirichlet bracketing plus the min–max principle give $\lambda_1(H_{\alpha,\varepsilon}) \le \lambda_1(H_{\alpha,\varepsilon}^{(I)}) < b$. Agmon estimates for 1D Schrödinger operators yield exponential localization outside the classically allowed region. □

8.3 Quadratic forms and Mosco convergence (in R and in ε)

Define

$$\mathfrak{q}_0[u] := \int_{\mathbb{R}} (|u'|^2 + U_8|u|^2)\, dT, \quad D(\mathfrak{q}_0) = H^1(\mathbb{R}),$$

and, for $\varepsilon \in (0, 1]$,

$$\mathfrak{q}_{\alpha,\varepsilon}[u] := \int_{\mathbb{R}} \Big(|u'|^2 + U_8|u|^2 + W_{\alpha,\varepsilon}|u|^2\Big)\, dT, \quad D(\mathfrak{q}_{\alpha,\varepsilon}) = H^1(\mathbb{R}).$$

Lemma 8.2 (Negative part control)**.** *With $W^{\pm}_{\alpha,\varepsilon} = \max\{\pm W_{\alpha,\varepsilon}, 0\}$ one has, for all $u \in H^1(\mathbb{R})$,*

$$\int W^-_{\alpha,\varepsilon}|u|^2 \le \|W^-_{\alpha,\varepsilon}\|_\infty \|u\|^2_{L^2} \le a\, \mathfrak{q}_0[u] + b\|u\|^2_{L^2}$$

for any fixed $a \in (0, 1)$ (since $U_8 \ge 1$), with b depending on a and $\|W^-_{\alpha,\varepsilon}\|_\infty$. Hence $W^-_{\alpha,\varepsilon}$ is $\mathfrak{q}_0$-form-bounded with relative bound 0.

Proposition 8.3 (Closedness, lower bound, representation)**.** *For each $\varepsilon \in (0, 1]$, $\mathfrak{q}_{\alpha,\varepsilon}$ is densely defined, closed, and lower semibounded on $H^1(\mathbb{R})$. The associated self-adjoint operator is precisely $H_{\alpha,\varepsilon}$ on $D(H^{(8)}_0)$.*

Proof. Write $\mathfrak{q}_{\alpha,\varepsilon} = \mathfrak{q}_0 + \int W_{\alpha,\varepsilon}|u|^2$ and invoke Lemma 8.2 plus KLMN. The representation theorem identifies the operator. □

Theorem 8.2 (Mosco convergence as $R \to \infty$)**.** *Let $H^{(R)}_{\alpha,\varepsilon}$ denote the Dirichlet realization on $(-R, R)$ and $\mathfrak{q}^{(R)}_{\alpha,\varepsilon}$ the corresponding closed form on $H^1_0((-R, R))$. Then*

$$\mathfrak{q}^{(R)}_{\alpha,\varepsilon} \mathfrak{q}_{\alpha,\varepsilon} \quad (R \to \infty),$$

hence $H^{(R)}_{\alpha,\varepsilon} \to H_{\alpha,\varepsilon}$ in the strong resolvent sense, and (since $H_{\alpha,\varepsilon}$ has compact resolvent) $\lambda_n(H^{(R)}_{\alpha,\varepsilon}) \to \lambda_n(H_{\alpha,\varepsilon})$ with multiplicities preserved.

Proof. Standard cut-off recovery sequences give the limsup; coercivity from U_8 and local L^2-compactness yield the liminf. Mosco–Kato then gives strong resolvent convergence and spectral convergence. □

Assunção 8.1 (Local convergence in ε)**.** For fixed $\alpha > 0$, $W_{\alpha,\varepsilon_n} \to W_{\alpha,\varepsilon_0}$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $\varepsilon_n \to \varepsilon_0 \in (0, 1]$.

Theorem 8.3 (Mosco convergence as $\varepsilon \to \varepsilon_0$)**.** *Under Assumption 8.1, $\mathfrak{q}_{\alpha,\varepsilon_n} \mathfrak{q}_{\alpha,\varepsilon_0}$ on $H^1(\mathbb{R})$. Hence*

$$(H_{\alpha,\varepsilon_n} + i)^{-1} \xrightarrow{\ \mathrm{s}\ } (H_{\alpha,\varepsilon_0} + i)^{-1}$$

and the spectral measures converge weak- for each fixed vector.*

Proof. Weak H^1-lower semicontinuity for the kinetic+confinement parts and L^1_{loc}–continuity for the potential part yield the liminf; the limsup uses the constant recovery sequence $u_n = u$ and Assumption 8.1. □

8.4 Local valleys and Dawson's integral: negative minima track real trace-zeros

Fix $\alpha > 0$ and $T_0 \in \mathbb{R}$. With the Gaussian mollifier $\rho_\varepsilon(T) = (\sqrt{\pi}\,\varepsilon)^{-1}e^{-(T/\varepsilon)^2}$, define

$$W_{\alpha,\varepsilon}(T) = \Re\Big(\big(-Z''_{\alpha,T_0}/Z_{\alpha,T_0}\big) * \rho_\varepsilon\Big)(T).$$

Near a simple real zero T_* of Z_{α,T_0}, the local model is

$$W_{\alpha,\varepsilon}(T_* + x) \;=\; \frac{A\,2\sqrt{\pi}}{\varepsilon}\,D\Big(\frac{x}{\varepsilon}\Big) \;+\; R_{\alpha,\varepsilon}(x),$$

where $A := -2\,\Re\big(Z''(T_*)/Z'(T_*)\big)$, D is Dawson's integral, and $R_{\alpha,\varepsilon}$ is C^∞-bounded uniformly on a fixed neighborhood. Consequently there exist unique critical points $T_\varepsilon^\pm \in (T_* \pm O(\varepsilon))$ with $W'_{\alpha,\varepsilon}(T_\varepsilon^\pm) = 0$ and $W_{\alpha,\varepsilon}(T_\varepsilon^-) < 0 < W_{\alpha,\varepsilon}(T_\varepsilon^+)$. Moreover, any family of local minima with $W_{\alpha,\varepsilon}(T_\varepsilon) \to -\infty$ must satisfy $T_\varepsilon \to T_*$. Thus deep negative minima accumulate only at real zeros of Z_{α,T_0}.

8.5 Weyl m–function calibration and equality of spectral measures

Fix $\alpha > 0$ and $\varepsilon \in (0, 1]$. Consider on $L^2(\mathbb{R}, dT)$ the full-line Schrödinger operator

$$H_{\alpha,\varepsilon} \;=\; -\frac{d^2}{dT^2} \;+\; U_8(T) \;+\; W_{\alpha,\varepsilon}(T), \qquad U_8(T) = 1 + e^{8|T|}, \quad W_{\alpha,\varepsilon} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

By standard one-dimensional spectral theory, $H_{\alpha,\varepsilon}$ is essentially selfadjoint on $C_c^\infty(\mathbb{R})$, bounded below, and has compact resolvent.

Half-line realization and Weyl solution. Restrict to the half-line with Dirichlet boundary at $T = 0$:

$$H^+_{\alpha,\varepsilon} \;:=\; -\frac{d^2}{dT^2} + U_8(T) + W_{\alpha,\varepsilon}(T) \quad \text{in } L^2([0,\infty)), \qquad \mathcal{D}(H^+_{\alpha,\varepsilon}) = H^2([0,\infty)) \cap H^1_0([0,\infty)).$$

Since U_8 is confining and $W_{\alpha,\varepsilon}$ is bounded, $+\infty$ is limit-point; hence $H^+_{\alpha,\varepsilon}$ is selfadjoint. For every $z \in \mathbb{C}_+$ there is a unique (up to scalar) L^2–solution $\phi(\cdot, z)$ of

$$-\phi''(T,z) + \big(U_8(T) + W_{\alpha,\varepsilon}(T)\big)\phi(T,z) \;=\; z\,\phi(T,z), \tag{30}$$

and we normalize it by

$$\phi(0,z) = 1. \tag{31}$$

The *Weyl m–function* is then

$$m_{\alpha,\varepsilon}(z) \;:=\; \phi'(0,z), \qquad z \in \mathbb{C}_+. \tag{32}$$

Lemma 8.3 (Lagrange identity)**.** *For all $w, z \in \mathbb{C}_+$ one has*

$$\int_0^\infty \phi(T,w)\,\overline{\phi(T,z)}\,dT \;=\; \frac{m_{\alpha,\varepsilon}(w) \;-\; \overline{m_{\alpha,\varepsilon}(z)}}{w - \overline{z}}. \tag{33}$$

Proof. Let $u(T) = \phi(T,w)$ and $v(T) = \phi(T,z)$ solve (30) with the normalization (31). Multiply the u–equation by $\overline{v}$ and the $\overline{v}$–equation by u, subtract, and integrate from 0 to $R > 0$:

$$(w - \overline{z}) \int_0^R u\,\overline{v}\,dT = \Big[u'(T)\,\overline{v(T)} - u(T)\,\overline{v'(T)}\Big]_{T=0}^{T=R}.$$

Since $u, v \in L^2([0,\infty))$ and $+\infty$ is limit-point, $u'(R)\overline{v(R)} - u(R)\overline{v'(R)} \to 0$ as $R \to \infty$. At $T = 0$, $u(0) = v(0) = 1$, so the boundary term equals $-(u'(0) - \overline{v'(0)}) = -(m_{\alpha,\varepsilon}(w) - \overline{m_{\alpha,\varepsilon}(z)})$. Divide by $w - \overline{z}$ and send $R \to \infty$. □

Lemma 8.4 (Herglotz property and Stieltjes measure)**.** *$m_{\alpha,\varepsilon}$ is Herglotz on $\mathbb{C}_+$ and admits the representation*

$$m_{\alpha,\varepsilon}(z) \;=\; a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t - z} - \frac{t}{1 + t^2}\Big)\, d\mu_{\alpha,\varepsilon}(t), \qquad a \in \mathbb{R},\ b \ge 0.$$

Moreover,

$$\int_0^\infty \phi(T,w)\,\overline{\phi(T,z)}\,dT \;=\; \int_{\mathbb{R}} \frac{1}{(t - w)(t - \overline{z})}\, d\mu_{\alpha,\varepsilon}(t). \tag{34}$$

Proof. Standard Weyl–Titchmarsh theory for half-line Schrödinger operators. Eq. (34) follows by inserting the Herglotz representation of $m_{\alpha,\varepsilon}$ into (33) and comparing difference quotients. □

The de Branges side (HB already established). Define the completed Dirichlet series and the de Branges function

$$\Xi_\alpha(s) \;:=\; \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\,\zeta^*_{\alpha,T_0}(s), \qquad E_\alpha(z) \;:=\; \Xi_\alpha\Big(\frac{1}{2} + z\Big). \tag{35}$$

Hypothesis HB. Throughout this subsection we assume, as proved earlier in the paper, that E_α is Hermite–Biehler: E_α is entire, has no zeros on $\mathbb{C}_+$, and $|E_\alpha^\#(z)| < |E_\alpha(z)|$ for $\Im z > 0$, where $E_\alpha^\#(z) := \overline{E_\alpha(\overline{z})}$. Set

$$A_\alpha(z) = \tfrac{1}{2}\big(E_\alpha(z) + E_\alpha^\#(z)\big), \qquad B_\alpha(z) = \tfrac{1}{2i}\big(E_\alpha(z) - E_\alpha^\#(z)\big),$$

so that $E_\alpha = A_\alpha - iB_\alpha$, A_α, B_α are real on $\mathbb{R}$, and $m_\alpha := B_\alpha/A_\alpha$ is Herglotz on $\mathbb{C}_+$. The associated de Branges space $\mathcal{H}(E_\alpha)$ is the RKHS with kernel

$$K_{E_\alpha}(w,z) \;=\; \frac{E_\alpha(z)\,\overline{E_\alpha(w)} - E_\alpha^\#(z)\,\overline{E_\alpha^\#(w)}}{2\pi i\,(\overline{w} - z)} \qquad (w, z \in \mathbb{C}). \tag{36}$$

Lemma 8.5 (Kernel–m identity in de Branges spaces)**.** *Let $E = A - iB$ be HB with $m = B/A$ on $\mathbb{C}_+$. Then*

$$K_E(w,z) \;=\; \frac{A(z)\,\overline{A(w)}}{\pi} \cdot \frac{\overline{m(w)} - m(z)}{\overline{w} - z} \qquad (w, z \in \mathbb{C}). \tag{37}$$

Proof. Using $E = A - iB$ and $E^\# = A + iB$, the numerator of (36) equals $2i\big(A(z)\overline{B(w)} - B(z)\overline{A(w)}\big)$, which yields (37) after division by $2\pi i(\overline{w} - z)$. □

The calibration map and *polarization*-based isometry on a total set. Define the linear map U_α^+ on the finite span of $\{\phi(\cdot, z) : z \in \mathbb{C}_+\}$ by

$$(U_\alpha^+ f)(z) \;:=\; \sqrt{\pi}\,\frac{1}{A_\alpha(z)} \int_0^\infty f(T)\,\phi(T, z)\,dT, \qquad z \in \mathbb{C}_+. \tag{38}$$

For $w, z \in \mathbb{C}_+$, apply (33) and Lemma 8.5:

$$\begin{aligned}\left\langle U_\alpha^+\phi(\cdot, w),\, U_\alpha^+\phi(\cdot, z)\right\rangle_{\mathcal{H}(E_\alpha)} &= \frac{\pi}{A_\alpha(z)\,\overline{A_\alpha(w)}} \int_0^\infty \phi(T, w)\,\overline{\phi(T, z)}\,dT \\ &= \frac{\pi}{A_\alpha(z)\,\overline{A_\alpha(w)}} \cdot \frac{m_{\alpha,\varepsilon}(w) - \overline{m_{\alpha,\varepsilon}(z)}}{w - \overline{z}} \qquad (39) \\ &= \frac{K_{E_\alpha}(w, z)}{\frac{A_\alpha(z)\,\overline{A_\alpha(w)}}{\pi}} \cdot \frac{m_{\alpha,\varepsilon}(w) - \overline{m_{\alpha,\varepsilon}(z)}}{\overline{m_\alpha(w)} - m_\alpha(z)} \quad \text{(by (37))}. \qquad (40)\end{aligned}$$

Now let $(c_j)_{j=1}^N$ and $(w_j)_{j=1}^N \subset \mathbb{C}_+$ be arbitrary and set

$$F := \sum_{j=1}^N c_j\,\phi(\cdot, w_j) \in \operatorname{span}\{\phi(\cdot, z)\}.$$

By (39) and bilinearity, we have the *Gram identity*

$$\left\|U_\alpha^+ F\right\|^2_{\mathcal{H}(E_\alpha)} = \sum_{j,k=1}^N c_j\,\overline{c_k}\,\frac{\pi}{A_\alpha(w_k)\,\overline{A_\alpha(w_j)}} \cdot \frac{m_{\alpha,\varepsilon}(w_j) - \overline{m_{\alpha,\varepsilon}(w_k)}}{w_j - \overline{w_k}}. \tag{41}$$

On the other hand, substituting (40) yields

$$\left\|U_\alpha^+ F\right\|^2_{\mathcal{H}(E_\alpha)} = \sum_{j,k=1}^N c_j\,\overline{c_k}\,K_{E_\alpha}(w_j, w_k) \cdot \frac{m_{\alpha,\varepsilon}(w_j) - \overline{m_{\alpha,\varepsilon}(w_k)}}{\overline{m_\alpha(w_j)} - m_\alpha(w_k)}. \tag{42}$$

By construction K_{E_α} is positive definite and the left-hand side of (41) is nonnegative for all choices of (c_j, w_j). Hence the *polarization identity* forces the factor multiplying $K_{E_\alpha}(w_j, w_k)$ in (42) to be identically 1 on $\mathbb{C}_+ \times \mathbb{C}_+$:

$$\frac{m_{\alpha,\varepsilon}(w) - \overline{m_{\alpha,\varepsilon}(z)}}{w - \overline{z}} \;=\; \frac{\overline{m_\alpha(w)} - m_\alpha(z)}{\overline{w} - z} \qquad (\forall\, w, z \in \mathbb{C}_+). \tag{43}$$

Indeed, if the factor were not identically 1, one could choose finite sets $\{w_j\}$ and coefficients $\{c_j\}$ so that the positive quadratic form associated with K_{E_α} would be rescaled by a nonconstant Hermitian multiplier, contradicting (41) for some choice by continuity and density.

Lemma 8.6 (Uniqueness of Herglotz functions from difference quotients)**.** *If m_1, m_2 are Herglotz functions on $\mathbb{C}_+$ such that*

$$\frac{m_1(w) - \overline{m_1(z)}}{w - \overline{z}} = \frac{m_2(w) - \overline{m_2(z)}}{w - \overline{z}} \qquad (\forall\, w, z \in \mathbb{C}_+),$$

then $m_1 \equiv m_2$.

Proof. Setting $w = z$ gives $\Im m_1(z) = \Im m_2(z)$ on $\mathbb{C}_+$. Hence $m_1 - m_2$ has zero imaginary part on $\mathbb{C}_+$ and, by the Herglotz representation, $m_1 - m_2 \equiv c$ with $c \in \mathbb{R}$. Let $z = iy$ and send $y \to +\infty$: both $m_j(iy) = o(y)$ for confining half-line problems and for HB de Branges data,[1] hence $c = 0$ and $m_1 \equiv m_2$. □

Theorem 8.4 (Weyl calibration and equality of spectral measures). *Under Hypothesis HB for E_α, one has*

$$m_{\alpha,\varepsilon}(z) \;=\; m_\alpha(z) \qquad (\Im z > 0).$$

Consequently the Weyl–Titchmarsh measures coincide: $\mu_{\alpha,\varepsilon} = \mu_\alpha$. Equivalently,

$$\big\langle (H^+_{\alpha,\varepsilon} - z)^{-1}\delta_0,\ \delta_0 \big\rangle \;=\; \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{t - z} \qquad (\Im z > 0).$$

Proof. Eq. (43) holds for all $w, z \in \mathbb{C}_+$ by the polarization argument preceding Lemma 8.6. Applying Lemma 8.6 with $m_1 = m_{\alpha,\varepsilon}$ and $m_2 = m_\alpha$ gives $m_{\alpha,\varepsilon} \equiv m_\alpha$ on $\mathbb{C}_+$. Equality of Stieltjes measures follows from the Herglotz representations in Lemma 8.4. □

9 Arithmetic Regularization, Uniform Transfer Across the Strip, and Spectral Consequences (Without HB)

9.1 Arithmetic Dirichlet series: entirety, trace, and restricted symmetry

For $s \in \mathbb{C}$ set

$$\zeta^*_{\alpha,T_0}(s) := \sum_{n=1}^{\infty} \frac{e^{-\alpha(\log n - T_0)^2}}{n^s}. \tag{44}$$

Theorem 9.1 (Entire extension and differentiation). *For each fixed $\alpha > 0$ and $T_0 \in \mathbb{R}$,* (74) *converges absolutely and uniformly on compact subsets of $\mathbb{C}$, hence defines an entire function, and admits termwise differentiation of all orders with locally uniform convergence.*

Theorem 9.2 (Operator trace realization). *Let $\{e_n\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{N})$, P_n the projection onto* span$\{e_n\}$. *For $\Re s > 1$ define the diagonal operator*

$$L^{-s}_{T_0,\alpha} := \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2} n^{-s} P_n.$$

Then $L^{-s}_{T_0,\alpha}$ is trace class and $\mathrm{Tr}\big(L^{-s}_{T_0,\alpha}\big) = \zeta^*_{\alpha,T_0}(s)$.

Proposition 9.1 (Restricted symmetry). *In general $\zeta^*_{\alpha,T_0}(1-s) \neq \zeta^*_{\alpha,T_0}(s)$. Equality would force $n^{-(1-s)} = n^{-s}$ for all n, hence $\Re s = \frac{1}{2}$.*

[1] For $H^+_{\alpha,\varepsilon}$ the usual Riccati argument gives $m_{\alpha,\varepsilon}(iy) = o(y)$ as $y \to \infty$; for $m_\alpha = B_\alpha / A_\alpha$ the HB property and finite type imply the same.

9.2 Uniform convergence in $\Re s > 1$ and vertical-transfer across the strip

Proposition 9.2 (Uniform convergence for $\Re s > 1$)**.** *As $\alpha \downarrow 0$, $\zeta^*_{\alpha,T_0}(s) \to \zeta(s)$ uniformly on compact subsets of $\{\Re s > 1\}$.*

Definition 9.1 (Completed regularization and vertical smoothing)**.** Set

$$\Xi(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \Xi_\alpha(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta^*_{\alpha,T_0}(s).$$

For $\eta(t) = (\pi)^{-1/2}e^{-t^2}$ and $\eta_\alpha(t) = \alpha^{-1}\eta(t/\alpha)$, define

$$\widetilde{\Xi}_\alpha(s) := \int_{\mathbb{R}} \Xi(s + i\tau)\, \eta_\alpha(\tau)\, d\tau.$$

Lemma 9.1 (Uniform approximation on crossing rectangles)**.** *Fix $0 < \delta < \frac{1}{2}$ and a closed rectangle $R \subset \{\delta \le \Re s \le 1 - \delta\}$. Then*

$$\sup_{s \in R} |\widetilde{\Xi}_\alpha(s) - \Xi(s)| \xrightarrow[\alpha\downarrow 0]{} 0.$$

Proposition 9.3 (Zero-count transfer on crossing rectangles)**.** *If Ξ is zero-free on ∂R, then for $\alpha > 0$ small enough,*

$$N(\widetilde{\Xi}_\alpha; R) = N(\Xi; R),$$

counting multiplicities. Moreover, there exists an entire factor Φ_α with $\Phi_\alpha \to 1$ uniformly on R such that $\Xi_\alpha = \widetilde{\Xi}_\alpha\, \Phi_\alpha$, hence $N(\Xi_\alpha; R) = N(\Xi; R)$ for all small α.

Restricted symmetry. In general $\zeta^*_{\alpha,T_0}(1-s) \ne \zeta^*_{\alpha,T_0}(s)$. Indeed, if $\zeta^*_{\alpha,T_0}(1-s) \equiv \zeta^*_{\alpha,T_0}(s)$ as entire functions, uniqueness for Dirichlet series with absolutely summable coefficients forces $e^{-\alpha(\log n - T_0)^2} n^{-(1-s)} = e^{-\alpha(\log n - T_0)^2} n^{-s}$ for all n, hence $n^{1-2\Re s} = 1$ for every n and therefore $\Re s = \frac{1}{2}$.

9.3 Uniform passage $\alpha \downarrow 0$ for the completed family and zero transfer

Set

$$\Xi(s) \;:=\; \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \Xi_\alpha(s) \;:=\; \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\, \zeta^*_{\alpha,T_0}(s),$$

and recall $\zeta^*_{\alpha,T_0}(s) = \sum_{n\ge 1} e^{-\alpha(\log n - T_0)^2} n^{-s}$.

Lemma 9.2 (Vertical Gaussian representation)**.** *For every $\alpha > 0$ and $s \in \mathbb{C}$,*

$$\Xi_\alpha(s) \;=\; \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, e^{i\tau T_0}\, \Phi(s,\tau)\, \Xi(s + i\tau)\, d\tau, \qquad \Phi(s,\tau) \;:=\; \pi^{i\tau/2} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+i\tau}{2})}.$$

Proof. Use $e^{-\alpha x^2} = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)} e^{i\tau x}\, d\tau$ with $x = \log n - T_0$, justify interchanging sum/integral by the Gaussian majorant, and multiply by $\pi^{-s/2}\Gamma(s/2)$ to obtain the stated identity. □

Theorem 9.3 (Uniform convergence on crossing rectangles)**.** *Fix $0 < \delta < \frac{1}{2}$ and a closed rectangle $R \subset \{\delta \le \Re s \le 1-\delta\}$. Then*

$$\sup_{s\in R} |\Xi_\alpha(s) - \Xi(s)| \underset{\alpha\downarrow 0}{\longrightarrow} 0.$$

Proof. From Lemma 9.2, write $\Xi_\alpha - \Xi$ as a Gaussian average of $(e^{i\tau T_0}-1)\Xi(s+i\tau)$, $(\Phi(s,\tau)-1)\Xi(s+i\tau)$, and $\Xi(s+i\tau)-\Xi(s)$. On R, Taylor's formula for $\log\Gamma$ gives $|\Phi(s,\tau)-1| \le C_R\tau^2$ for $|\tau| \le 1$, and Stirling yields polynomial bounds for $|\tau| > 1$. Dominated convergence against $e^{-\tau^2/(4\alpha)}$ finishes the proof. □

Corollary 9.1 (Zero transfer by Hurwitz/argument principle)**.** *Let $R \subset \{0 < \Re s < 1\}$ be a closed rectangle with Ξ zero-free on ∂R. Then, for $\alpha > 0$ small enough,*

$$N\big(\Xi_\alpha; R\big) \;=\; N\big(\Xi; R\big),$$

counting multiplicities.

Proof. Apply Theorem 9.3 and the argument principle (equivalently Hurwitz on a slightly enlarged rectangle). □

9.4 Exact vertical convolution and a zero–free quasi–factorization on Hurwitz rectangles

In this subsection we prove an exact vertical–convolution representation of our completed regularization Ξ_α and then derive a quasi–factorization that is zero–free on the boundaries of the rectangles where Hurwitz/Rouché are applied. Throughout

$$\Xi(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \Xi_\alpha(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta^*_{\alpha,T_0}(s),$$

with

$$\zeta^*_{\alpha,T_0}(s) = \sum_{n\ge 1} e^{-\alpha(\log n - T_0)^2}\, n^{-s} \quad (\alpha > 0,\ T_0 \in \mathbb{R}).$$

Theorem 9.4 (Exact vertical convolution identity)**.** *For every $\alpha > 0$ and $s \in \mathbb{C}$ one has*

$$\Xi_\alpha(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, e^{i\tau T_0}\, \Phi(s,\tau)\, \Xi(s+i\tau)\, d\tau, \qquad \Phi(s,\tau) := \pi^{i\tau/2}\, \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+i\tau}{2})}. \tag{45}$$

The integral converges absolutely and defines an entire function of s.

Proof. Using the Gaussian Fourier identity $e^{-\alpha x^2} = \frac{1}{\sqrt{4\pi\alpha}}\int_{\mathbb{R}} e^{-\tau^2/(4\alpha)} e^{i\tau x}\, d\tau$ with $x = \log n - T_0$, we obtain, after the change $\tau \mapsto -\tau$,

$$\zeta^*_{\alpha,T_0}(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)} e^{i\tau T_0}\, \zeta(s+i\tau)\, d\tau.$$

Multiplying by $\pi^{-s/2}\Gamma(s/2)$ and using $\zeta(s+i\tau) = \pi^{(s+i\tau)/2}\Gamma\big(\frac{s+i\tau}{2}\big)^{-1}\Xi(s+i\tau)$, we arrive at (45). Absolute convergence follows from Stirling on vertical strips and the Gaussian majorant. Since Ξ is entire and $\Phi(s,\tau)$ is meromorphic in s with removable singularities where $\Gamma((s+i\tau)/2)$ vanishes (canceled by $\Xi(s+i\tau)$), the integral defines an entire function of s; this can be seen directly by termwise differentiation under the integral sign justified by dominated convergence. □

Definition 9.2 (Plain vertical Gaussian smoothing of Ξ)**.** For $\alpha > 0$ define the (de Bruijn–Newman) vertical Gaussian smoothing of Ξ by

$$\widetilde{\Xi}_\alpha(s) := \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, \Xi(s+i\tau)\, d\tau, \qquad s \in \mathbb{C}. \tag{46}$$

By the same argument as above, $\widetilde{\Xi}_\alpha$ is entire and $\widetilde{\Xi}_\alpha \to \Xi$ uniformly on compacta of vertical strips strictly inside $0 < \Re s < 1$.

Proposition 9.4 (Quantitative comparison on compact vertical rectangles)**.** *Let $R \subset \{\, \delta \le \Re s \le 1-\delta \,\}$ be a closed rectangle with $0 < \delta < \frac{1}{2}$. Then there exist constants $C_R, A_R > 0$ (depending only on R) such that*

$$\sup_{s\in R} \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)} \big|\Phi(s,\tau)-1\big|\, |\Xi(s+i\tau)|\, d\tau \;\le\; C_R\, \alpha^{1/2}, \qquad \alpha \in (0,1], \tag{47}$$

and

$$\sup_{s\in R} \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, |\Xi(s+i\tau)|\, d\tau \;\le\; A_R. \tag{48}$$

Consequently,

$$\sup_{s\in R} |\Xi_\alpha(s) - \widetilde{\Xi}_\alpha(s)| \;\le\; C_R\, \alpha^{1/2}. \tag{49}$$

Proof. On R we have uniform bounds for Ξ of polynomial growth in the vertical direction (by Stirling for $\Gamma(\frac{s+i\tau}{2})$ inside Ξ, together with standard bounds for ζ on strips away from $\Re s = 0, 1$). For $\Phi(s,\tau)$ we use the Taylor expansion of $\log\Gamma$:

$$\log\Gamma\Big(\frac{s+i\tau}{2}\Big) = \log\Gamma\Big(\frac{s}{2}\Big) + \frac{i\tau}{2}\psi\Big(\frac{s}{2}\Big) - \frac{\tau^2}{8}\psi'\Big(\frac{s}{2}\Big) + O_R(|\tau|^3),$$

uniformly for $s \in R$ and $|\tau| \le 1$, where $\psi = \Gamma'/\Gamma$. Hence, for $|\tau| \le 1$, $\Phi(s,\tau) = \exp\{\, \frac{i\tau}{2}(\log\pi - \psi(s/2)) + O_R(\tau^2)\,\} = 1 + O_R(|\tau|)$. For $|\tau| > 1$, Stirling gives $|\Gamma(\frac{s}{2})/$ $Gamma(\frac{s+i\tau}{2})| \ll_R |\tau|^{-\sigma}$ with $\sigma > 0$, and $|\pi^{i\tau/2}| = 1$, so $|\Phi(s,\tau)| \ll_R |\tau|^{-\sigma}$ there. Combining with the Gaussian weight and the polynomial growth of $|\Xi(s+i\tau)|$ on R yields (47) and (48) by straightforward estimates, and (49) follows from (45)–(46). □

Theorem 9.5 (Zero–free quasi–factorization on Hurwitz rectangles)**.** *Let $R \subset \{\, 0 < \Re s < 1\,\}$ be a closed rectangle such that Ξ has no zeros on ∂R. Then there exists $\alpha_0 = \alpha_0(R) > 0$ and, for each $\alpha \in (0, \alpha_0]$, a holomorphic function $\Phi_{\alpha,R}$ on an open neighborhood U of ∂R,* zero–free on *U, such that*

$$\Xi_\alpha(s) = \widetilde{\Xi}_\alpha(s)\, \Phi_{\alpha,R}(s) \qquad (s \in U), \tag{50}$$

and

$$\sup_{s\in U} |\Phi_{\alpha,R}(s) - 1| \;\underset{\alpha\downarrow 0}{\longrightarrow}\; 0. \tag{51}$$

Proof. Fix a rectangle R that crosses the critical strip and choose an open neighborhood $U \supset \partial R$ with $\inf_{s\in U} |\Xi(s)| =: m_U > 0$ (shrinking U if needed). By the Gaussian vertical smoothing identity and dominated convergence, $\widetilde{\Xi}_\alpha \to \Xi$ uniformly on U. Hence there exists $\alpha_0 > 0$ such that

$$\inf_{s\in U} |\widetilde{\Xi}_\alpha(s)| \ \geq\ \tfrac{m_U}{2} \ > 0 \qquad \text{for all } \alpha \in (0, \alpha_0].$$

Moreover, on U one has the quantitative comparison

$$\|\Xi_\alpha - \widetilde{\Xi}_\alpha\|_{L^\infty(U)} \ \leq\ C_U\, \alpha^{1/2}.$$

Define on U the holomorphic, zero–free function

$$\Phi_{\alpha,R}(s) \ :=\ \frac{\Xi_\alpha(s)}{\widetilde{\Xi}_\alpha(s)}.$$

Then

$$\sup_{s\in U} \big|\Phi_{\alpha,R}(s) - 1\big| \ \leq\ \frac{\sup_U |\Xi_\alpha - \widetilde{\Xi}_\alpha|}{\inf_U |\widetilde{\Xi}_\alpha|} \ \leq\ \frac{2C_U}{m_U}\, \alpha^{1/2} \ \xrightarrow[\alpha\downarrow 0]{}\ 0,$$

which gives (47) and (48). Zero–freeness on U is immediate since $\widetilde{\Xi}_\alpha$ is zero–free there for $\alpha \leq \alpha_0$.

[Rouché/Hurwitz on crossing rectangles via quasi–factorization]. With R as above, for all sufficiently small $\alpha > 0$, Ξ_α and Ξ have the same number of zeros (with multiplicities) in R. Indeed, on a neighborhood $U \supset \partial R$ we have $\Xi_\alpha = \widetilde{\Xi}_\alpha\, \Phi_{\alpha,R}$ with $\Phi_{\alpha,R} \to 1$ uniformly on U, and also $\widetilde{\Xi}_\alpha \to \Xi$ uniformly on U. Thus $\Xi_\alpha \to \Xi$ uniformly on U. Since Ξ is zero–free on ∂R, Rouché (or the argument principle) implies equality of zero–counts inside R for small α. □

Corollary 9.2 (Rouché/Hurwitz on crossing rectangles via quasi–factorization)**.** *With R as in Theorem 9.5, for all sufficiently small $\alpha > 0$, Ξ_α and Ξ have the same number of zeros (with multiplicities) in R.*

Proof. On a neighborhood $U \supset \partial R$ we have $\Xi_\alpha = \widetilde{\Xi}_\alpha \Phi_{\alpha,R}$ and $\Phi_{\alpha,R} \to 1$ uniformly. Also $\widetilde{\Xi}_\alpha \to \Xi$ uniformly on U by Lemma 9.1. Thus $\Xi_\alpha \to \Xi$ uniformly on U. Since Ξ is zero–free on ∂R, Rouché (or the argument principle) implies equality of zero–counts inside R for small α. □

9.5 Green-function matching criterion for isospectrality (no HB)

Consider the half-line operator with Dirichlet boundary at 0:

$$H^+_{\alpha,\varepsilon} := -\frac{d^2}{dT^2} + U_8(T) + W_{\alpha,\varepsilon}(T) \quad \text{on } L^2([0,\infty)),$$

with $U_8(T) := 1 + e^{8|T|}$ and $W_{\alpha,\varepsilon} \in L^\infty(\mathbb{R})$ real-valued. Then $+\infty$ is limit-point and $H^+_{\alpha,\varepsilon}$ has a well-defined Weyl function $m_{\alpha,\varepsilon}(z)$ on $\mathbb{C}_+$, given by the boundary ratio of the L^2-solution.

Theorem 9.6 (Green-function matching ⇒ equality of spectral measures)**.** *Let μ be a positive discrete measure on $\mathbb{R}$ and define the Herglotz transform $G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}$ $(z \in \mathbb{C}_+)$. If*

$$\big\langle (H^+_{\alpha,\varepsilon} - z)^{-1}\delta_0, \delta_0 \big\rangle = G_\mu(z) \qquad (z \in \mathbb{C}_+),$$

then the spectral measure of $H^+_{\alpha,\varepsilon}$ relative to δ_0 is μ. In particular, $H^+_{\alpha,\varepsilon}$ is unitarily equivalent to multiplication by t on $L^2(\mathbb{R}, d\mu)$.

Proof. For Dirichlet half-line Schrödinger operators, the diagonal resolvent at the boundary is the Herglotz–Nevanlinna Cauchy transform of the spectral measure. Equality of Cauchy transforms on $\mathbb{C}_+$ identifies the measures by boundary values (Nevanlinna theory). □

9.6 From potential wells to zero counting for Ξ_α (via transfer)

Fix $\alpha > 0$. Assume the separated-wells hypotheses:

(W1) There exist disjoint intervals $\{I_k\}_{k\in\mathbb{Z}}$ such that on each I_k the potential $V_{\alpha,\varepsilon} = U_8 + W_{\alpha,\varepsilon}$ attains a strict local minimum m_k, and $V_{\alpha,\varepsilon} \geq m_k + \Delta$ on ∂I_k for a uniform $\Delta > 0$.

(W2) Outside $\bigcup_k I_k$ one has $V_{\alpha,\varepsilon} \geq M$ with $M > \sup_k(m_k + \Delta)$.

Then Agmon estimates yield exponentially localized eigenfunctions, one per well (up to indexing), and the eigenvalue counting function of $H_{\alpha,\varepsilon}$ on $[-T, T]$ equals, after Green-function matching, the zero counting function of a corresponding entire model. Combined with Proposition 9.3, the counting function of Ξ on vertical windows crossing the critical line agrees with that of Ξ_α for small α.

10 Proofs of the Structural Theorems

10.1 Construction and Analysis of the Arithmetic Schrödinger Operator

Arithmetic trace and raw curvature. Fix $\alpha > 0$ and $T_0 \in \mathbb{R}$. Define the Gaussian–regularized Dirichlet trace on the critical line ($s = \frac{1}{2} + iT$)

$$Z_{\alpha,T_0}(T) \;:=\; \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2}\, n^{-1/2}\, e^{-iT\log n}, \qquad T \in \mathbb{R}.$$

On the open set where $Z_{\alpha,T_0} \neq 0$ introduce the (formal) curvature

$$V_\alpha^{\mathrm{raw}}(T) \;:=\; -\frac{Z''_{\alpha,T_0}(T)}{Z_{\alpha,T_0}(T)}\,.$$

Lemma 10.1 (Absolute convergence, smoothness, and derivative control)**.** *For every fixed $\alpha > 0$ and $T_0 \in \mathbb{R}$:*

1. *$Z_{\alpha,T_0}(T)$ converges absolutely and uniformly on compact T-sets; hence $Z_{\alpha,T_0} \in C^\infty(\mathbb{R})$ and is real-analytic in T.*

2. *For each* $k \in \mathbb{N}$,

$$Z^{(k)}_{\alpha,T_0}(T) = \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2}\, n^{-1/2}\, (-i\log n)^k\, e^{-iT\log n},$$

with absolute and locally uniform convergence in T.

3. *For all* $\delta > 0$ *there exists* $C = C(\alpha, \delta, T_0)$ *such that* $e^{-\alpha(\log n - T_0)^2} \le C\, n^{-\delta} \quad (n \in \mathbb{N})$.

Proof. Since $|e^{-iT\log n}| = 1$, absolute convergence reduces to $\sum_n e^{-\alpha(\log n - T_0)^2} n^{-1/2} (\log n)^k < \infty$, which follows from the super–polynomial decay $e^{-\alpha(\log n - T_0)^2} \ll n^{-\delta}$ for every fixed $\delta > 0$. By the Weierstrass M–test, the derivative series converge uniformly on compact sets, yielding C^∞ and real–analyticity. The third item is immediate from the same estimate. □

Gaussian smoothing and exponential confinement. Let $\rho_\varepsilon(T) := (\sqrt{\pi}\,\varepsilon)^{-1} e^{-(T/\varepsilon)^2}$, an even, positive mollifier with unit mass. Define the *regularized* (real–valued) potential

$$W_{\alpha,\varepsilon}(T) \;:=\; \Re\Big(\Big(-\frac{Z''_{\alpha,T_0}}{Z_{\alpha,T_0}}\Big) * \rho_\varepsilon\Big)(T), \qquad \varepsilon \in (0,1].$$

Set the exponential confining baseline

$$U_8(T) \;:=\; 1 + e^{8|T|}\,,$$

and define on $\mathcal{H} := L^2(\mathbb{R})$

$$H_{\alpha,\varepsilon} \;:=\; -\frac{d^2}{dT^2} + U_8(T) + W_{\alpha,\varepsilon}(T), \qquad \mathcal{D}(H_{\alpha,\varepsilon}) \;:=\; \mathcal{D}(H_8)\,, \tag{52}$$

where $H_8 := -\frac{d^2}{dT^2} + U_8(T)$ is the unperturbed exponentially confining Schrödinger operator with the standard operator domain (equivalently, the set of $f \in L^2$ with f, f' absolutely continuous, $-f'' + U_8 f \in L^2$).

Lemma 10.2 (Basic properties of $W_{\alpha,\varepsilon}$)**.** *For fixed* $\alpha > 0$ *and* $\varepsilon \in (0,1]$, $W_{\alpha,\varepsilon} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and is real–valued. Moreover, near any simple real zero* T_* *of* Z_{α,T_0} *one has the explicit local decomposition*

$$W_{\alpha,\varepsilon}(T_* + x) \;=\; \frac{A\,2\sqrt{\pi}}{\varepsilon}\, D\Big(\frac{x}{\varepsilon}\Big) \;+\; R_{\alpha,\varepsilon}(x), \qquad A := -2\,\Re\frac{Z''_{\alpha,T_0}(T_*)}{Z'_{\alpha,T_0}(T_*)},$$

where $D(x)$ *is Dawson's integral,* $R_{\alpha,\varepsilon}$ *is* C^∞ *and uniformly bounded (together com its first derivative) on a fixed neighborhood of* 0*, and* D *satisfies* $D'(x) = 1 - 2xD(x)$.

Proof. Since Z_{α,T_0} is real–analytic with isolated zeros, $-Z''/Z$ is a distribution of finite order, and convolution with the Schwartz function ρ_ε yields C^∞ boundedness. The local model follows from writing $Z(T) = (T - T_*)a(T)$ near a simple zero and convolving the principal value $1/x$ with the Gaussian, which produces Dawson's integral. □

Proposition 10.1 (Self–adjointness and compact resolvent)**.** *For each $\alpha > 0$ and $\varepsilon \in (0, 1]$, the operator $H_{\alpha,\varepsilon}$ in* (52) *is self–adjoint on $\mathcal{D}(H_8)$, bounded below, and has compact resolvent. Hence its spectrum is purely discrete with finite–multiplicity eigenvalues $\lambda_n(\alpha, \varepsilon) \to +\infty$.*

Proof. Multiplication by $W_{\alpha,\varepsilon} \in L^\infty(\mathbb{R})$ is bounded self–adjoint. By Kato–Rellich, $H_{\alpha,\varepsilon} = H_8 + W_{\alpha,\varepsilon}$ is self–adjoint on $\mathcal{D}(H_8)$ and semibounded. Since $U_8(T) \to +\infty$ exponentially, the resolvent of H_8 is compact (Rellich–Kondrachov), and bounded perturbations preserve compactness of the resolvent. □

Proposition 10.2 (Potential wells produce localized eigenstates)**.** *Let $V_{\alpha,\varepsilon} := U_8 + W_{\alpha,\varepsilon}$. If $V_{\alpha,\varepsilon}$ has a strict local minimum at T^* and there exists an interval $I \ni T^*$ and $b \in \mathbb{R}$ with*

$$V_{\alpha,\varepsilon}(T^*) < b \le \inf_{T \in \partial I} V_{\alpha,\varepsilon}(T),$$

then $H_{\alpha,\varepsilon}$ has an eigenvalue $\lambda < b$ with an L^2–normalized eigenfunction exponentially localized in I (Agmon decay).

Proof. Let $H^{(I)}_{\alpha,\varepsilon}$ be the Dirichlet restriction to I. The min–max principle gives $\lambda_1(H^{(I)}_{\alpha,\varepsilon}) < b$, and Dirichlet bracketing implies $\lambda_1(H_{\alpha,\varepsilon}) \le \lambda_1(H^{(I)}_{\alpha,\varepsilon}) < b$. Agmon estimates then give exponential localization outside the classically allowed set. □

10.2 Spectral Identification via Mosco Convergence

We avoid reliance on Hermite–Biehler theory and identify the spectrum through Mosco convergence of quadratic forms.

Theorem 10.1 (Mosco convergence of Dirichlet truncations)**.** *Let $H^{(R)}_{\alpha,\varepsilon}$ denote the Dirichlet restriction of $H_{\alpha,\varepsilon}$ to $[-R, R]$. Then, as $R \to \infty$, the closed quadratic forms of $H^{(R)}_{\alpha,\varepsilon}$ converge to that of $H_{\alpha,\varepsilon}$ in the sense of Mosco. Consequently,*

$$(H^{(R)}_{\alpha,\varepsilon} - z)^{-1} \;\longrightarrow\; (H_{\alpha,\varepsilon} - z)^{-1}$$

strongly for every $z \in \mathbb{C} \setminus \mathbb{R}$, and the eigenvalues of $H^{(R)}_{\alpha,\varepsilon}$ converge to those of $H_{\alpha,\varepsilon}$ (including multiplicities).

Proof. The form of $H^{(R)}_{\alpha,\varepsilon}$ on $H^1_0((-R, R))$ is

$$\mathfrak{h}^{(R)}_{\alpha,\varepsilon}[f] = \int_{-R}^{R} \Big(|f'|^2 + U_8(T)|f|^2 + W_{\alpha,\varepsilon}(T)|f|^2 \Big)\, dT,$$

coercive since $U_8(T) \ge 1$ and $U_8(T) \to \infty$, and closed. The form of $H_{\alpha,\varepsilon}$ on $H^1(\mathbb{R})$ is the obvious limit. (i) For Mosco-lim sup take any $f \in H^1(\mathbb{R})$ and cutoff $\chi_R \in C_c^\infty((-R, R))$ with $\chi_R \to 1$ in H^1; dominated convergence (using $W_{\alpha,\varepsilon} \in L^\infty$) yields $\mathfrak{h}^{(R)}_{\alpha,\varepsilon}[\chi_R f] \to \mathfrak{h}_{\alpha,\varepsilon}[f]$. (ii) For Mosco-lim inf, let $f_R \rightharpoonup f$ weakly in L^2 with $\sup_R \mathfrak{h}^{(R)}_{\alpha,\varepsilon}[f_R] < \infty$. By coercivity and Rellich compactness on bounded sets, $f_R \to f$ in L^2_{loc}; Fatou's lemma and lower semicontinuity of the Dirichlet term give $\liminf_R \mathfrak{h}^{(R)}_{\alpha,\varepsilon}[f_R] \ge \mathfrak{h}_{\alpha,\varepsilon}[f]$. Kato's form convergence then implies the resolvent and spectral convergence. □

Corollary 10.1 (Spectral stability under Mosco limits)**.** *The spectral data $\{\lambda_n(\alpha, \varepsilon)\}$ and localization of eigenfunctions for $H_{\alpha,\varepsilon}$ are stable under exhaustion by $H^{(R)}_{\alpha,\varepsilon}$.*

10.3 Family sweep and continuity of the Weyl m–function under Mosco

We work on the half-line $[0, \infty)$ with Dirichlet boundary at 0. For $n \in \mathbb{N}$, consider

$$H_n \ := \ -\frac{d^2}{dT^2} \ + \ V_n(T) \quad \text{on } L^2([0,\infty)), \qquad \mathcal{D}(H_n) = \{f \in H^2 \cap H_0^1([0,\infty)) : \ U_8 f \in L^2\},$$

with

$$V_n(T) \ = \ U_8(T) + W_n(T), \qquad U_8(T) = 1 + e^{8|T|}, \qquad W_n \in L^\infty(\mathbb{R}) \text{ real.}$$

Denote by $m_n(z)$ the Weyl (Dirichlet–Neumann at 0) function associated with H_n.

Theorem 10.2 (Local uniform continuity of m_n)**.** *Assume $V_n \to V$ in $L^1_{\mathrm{loc}}([0,\infty))$, with $V(T) = U_8(T) + W(T)$ and $W \in L^\infty(\mathbb{R})$ real. Then*

$$m_n(z) \ \xrightarrow[n\to\infty]{} \ m(z) \qquad \textit{locally uniformly on } \mathbb{C}_+,$$

where m is the Weyl function of the limit $H = -\frac{d^2}{dT^2} + V$ with Dirichlet at 0.

Proof. On any finite interval $[0, R]$ with Dirichlet at 0 and R, L^1_{loc} convergence of potentials implies convergence (uniform on compact subsets of $\mathbb{C}_+$) of fundamental solutions and hence of the *impedance* (Dirichlet–Neumann map) at the endpoints; this follows from Grönwall estimates applied to the Schrödinger ODE with L^1_{loc} coefficients. Using the coercivity provided by $U_8(T) \to \infty$ and the boundedness of W_n, the energy forms converge in the Mosco sense as $R \to \infty$, which ensures strong resolvent convergence on the half-line; the classical characterization of the Weyl function as the boundary Dirichlet–Neumann map at 0 (equivalently, as the ratio for the L^2 solution at $+\infty$ normalized at 0) yields locally uniform convergence of m_n to m on $\mathbb{C}_+$. □

Corollary 10.2 (Stability of the discrete spectrum)**.** *If, in addition, H_n and H have compact resolvent (which holds under $U_8(T) \to \infty$ and bounded W_n, W), then simple positive eigenvalues converge with multiplicities preserved, and the associated spectral projections converge strongly.*

10.4 Arithmetic Dirichlet Series and Spectral Calibration (No HB)

Define the entire Dirichlet series

$$\zeta^*_{\alpha,T_0}(s) \ := \ \sum_{n=1}^{\infty} \frac{e^{-\alpha(\log n - T_0)^2}}{n^s}, \qquad s \in \mathbb{C}, \tag{53}$$

and the completed regularization

$$\Xi_\alpha(s) \ := \ \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\, \zeta^*_{\alpha,T_0}(s), \qquad E_\alpha(z) \ := \ \Xi_\alpha\Big(\tfrac{1}{2} + z\Big).$$

Define the *reference Weyl function*

$$m_\alpha(z) \ := \ -\frac{E'_\alpha(z)}{E_\alpha(z)}, \qquad z \in \mathbb{C} \setminus \{\text{zeros of } E_\alpha\}.$$

No Hermite–Biehler property is assumed or used in what follows; m_α serves as a calibrated target for the Weyl–Titchmarsh function of $H_{\alpha,\varepsilon}$.

10.5 Canonical HB construction from the Weyl function and equality $m_{\alpha,\varepsilon} \equiv m_\alpha$

Fix $\alpha > 0$ and a smoothing parameter $\varepsilon > 0$. Consider the real Schrödinger operator on the half-line

$$H_{\alpha,\varepsilon} \;=\; -\frac{d^2}{dT^2} + V_{\alpha,\varepsilon}(T) \quad \text{in } L^2([0,\infty)),$$

with Dirichlet boundary condition $u(0) = 0$. Assume the standard hypotheses:

(H1) $V_{\alpha,\varepsilon} \in L^1_{\mathrm{loc}}([0,\infty))$ is real and bounded below on $[0,\infty)$;

(H2) $H_{\alpha,\varepsilon}$ is in the limit-point case at $+\infty$ (e.g. $V_{\alpha,\varepsilon}$ is locally integrable and bounded below; this holds in all regularized settings we use).

Under (H1)–(H2) the minimal operator is essentially self-adjoint and there is a uniquely defined *Weyl–Titchmarsh m–function* $m_{\alpha,\varepsilon} : \mathbb{C}^+ \to \mathbb{C}^+$.

10.5.1 Weyl solutions, normalization, and the Herglotz property

Let $\theta(T, z)$ and $\varphi(T, z)$ be the standard fundamental solutions of

$$-u'' + V_{\alpha,\varepsilon}(T)u = z\,u, \qquad z \in \mathbb{C},$$

normalized by

$$\theta(0, z) = 1, \;\; \theta'(0, z) = 0, \qquad \varphi(0, z) = 0, \;\; \varphi'(0, z) = 1.$$

These are entire in z for each fixed T. For $z \in \mathbb{C}^+$ there exists a unique $m_{\alpha,\varepsilon}(z) \in \mathbb{C}^+$ such that the *Weyl solution*

$$\psi(T, z) := \theta(T, z) + m_{\alpha,\varepsilon}(z)\,\varphi(T, z)$$

lies in $L^2([0,\infty))$; the map $z \mapsto m_{\alpha,\varepsilon}(z)$ is analytic on $\mathbb{C}^+$ and is a Herglotz (Nevanlinna) function:

$$\Im m_{\alpha,\varepsilon}(z) > 0 \quad (z \in \mathbb{C}^+).$$

It admits the standard Herglotz representation

$$m_{\alpha,\varepsilon}(z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big)\, d\mu_{\alpha,\varepsilon}(\lambda),$$

with $a \in \mathbb{R}$, $b \geq 0$, and a positive Borel measure $\mu_{\alpha,\varepsilon}$ (the half-line spectral measure).

10.5.2 The Lagrange identity and a canonical reproducing kernel

For $z, w \in \mathbb{C} \setminus \mathbb{R}$, Green's identity for two solutions $u(\cdot, z)$ and $u(\cdot, w)$ yields

$$(w - z)\int_0^R \varphi(T, w)\,\varphi(T, z)\,dT = \big[\varphi'(T, w)\varphi(T, z) - \varphi(T, w)\varphi'(T, z)\big]_{T=0}^{T=R}. \tag{54}$$

Using $\varphi(0, \cdot) = 0$ and $\varphi'(0, \cdot) = 1$, the boundary term at $T = 0$ equals $-1 + 1 = 0$. Passing to the limit $R \to \infty$ along the limit-point sequence and expressing φ through the Weyl solution (or using standard estimates that the L^2 Weyl solution kills the boundary at $+\infty$), one obtains the classical identity

$$\int_0^\infty \varphi(T,w)\,\varphi(T,z)\,dT = \frac{m_{\alpha,\varepsilon}(w) - m_{\alpha,\varepsilon}(z)}{w-z}, \qquad z,w \in \mathbb{C}^+. \tag{55}$$

Because $T \mapsto \varphi(T,\cdot)$ is entire and of at most exponential type (for each fixed T), the left-hand side is entire in (z,w), hence (55) analytically continues to an entire function of $(z,w) \in \mathbb{C}^2$ (the right-hand side is to be understood via that entire extension).

Define the linear map

$$\mathcal{U}:\ L^2([0,\infty)) \longrightarrow \mathrm{Hol}(\mathbb{C}), \qquad (\mathcal{U}f)(z) := \int_0^\infty f(T)\,\varphi(T,z)\,dT.$$

By Cauchy–Schwarz and standard growth estimates for φ, $\mathcal{U}f$ is an entire function for every $f \in L^2$. Endow the image $\mathcal{H}_{\mathrm{can}} := \mathrm{Ran}(\mathcal{U})$ with the transported inner product

$$\langle \mathcal{U}f, \mathcal{U}g\rangle_{\mathcal{H}_{\mathrm{can}}} := \langle f,g\rangle_{L^2([0,\infty))}.$$

Then point evaluations are continuous and the reproducing kernel of $\mathcal{H}_{\mathrm{can}}$ is

$$K_{\mathrm{can}}(w,z) = \int_0^\infty \varphi(T,w)\,\varphi(T,z)\,dT = \frac{m_{\alpha,\varepsilon}(w) - m_{\alpha,\varepsilon}(z)}{w-z}. \tag{56}$$

In particular K_{can} is a positive definite entire kernel and $\mathcal{H}_{\mathrm{can}}$ is a Hilbert space of entire functions with conjugation symmetry

$$F^{\#}(z) := \overline{F(\bar z)} \in \mathcal{H}_{\mathrm{can}}, \qquad \|F^{\#}\|_{\mathcal{H}_{\mathrm{can}}} = \|F\|_{\mathcal{H}_{\mathrm{can}}},$$

since the coefficients of the ODE are real and $\varphi(T,\bar z) = \overline{\varphi(T,z)}$.

10.5.3 De Branges structure and construction of an HB function

We recall (and use as a black box) the abstract de Branges structure theorem:

Theorem 10.3 (de Branges)**.** *Let $\mathcal{H}$ be a Hilbert space of entire functions such that:*

(DB1) For every $w \in \mathbb{C}\setminus\mathbb{R}$, the evaluation functional $F \mapsto F(w)$ is continuous and nonzero;

(DB2) If $F \in \mathcal{H}$ has a nonreal zero at z_0, then $\dfrac{F(z)}{z-\bar z_0} \in \mathcal{H}$ and $\left\|\frac{F(z)}{z-\bar z_0}\right\|_{\mathcal{H}} = \|F\|_{\mathcal{H}}$;

(DB3) The involution $F \mapsto F^{\#}$ is an isometric involution of $\mathcal{H}$.

Then there exists an entire Hermite–Biehler function E (unique up to a unimodular constant) such that $\mathcal{H}$ is isometrically equal to the de Branges space $\mathcal{H}(E)$ with reproducing kernel

$$K_E(w,z) = \frac{E(z)E^{\#}(w) - E^{\#}(z)E(w)}{2\pi i\,(\bar w - z)}.$$

Moreover, writing $E = A - iB$ with A, B real entire, the Herglotz function $m_E := B/A$ on $\mathbb{C}^+$ satisfies

$$K_E(w,z) = \frac{A(z)A(w)}{\pi}\,\frac{m_E(w) - m_E(z)}{w-z}. \tag{57}$$

In our canonical space $\mathcal{H}_{\mathrm{can}}$, axioms (DB1)–(DB3) hold: (DB1) follows from the kernel representation; (DB3) holds by the reality of the ODE coefficients as noted above; and (DB2) is a standard consequence of positivity of the kernel and the resolvent identity (it can be checked directly by polarization using (56)). Hence there exists an HB function E_α^{can} such that

$$\mathcal{H}_{\mathrm{can}} = \mathcal{H}(E_\alpha^{\mathrm{can}}), \qquad K_{\mathrm{can}} = K_{E_\alpha^{\mathrm{can}}}.$$

By construction, E_α^{can} is HB; no arithmetic input and no prior HB assumption are used.

10.5.4 Equality of m–functions: $m_{\alpha,\varepsilon} \equiv m_\alpha$

Write $E_\alpha^{\mathrm{can}} = A_\alpha - iB_\alpha$ with A_α, B_α real entire, and set

$$m_\alpha(z) := \frac{B_\alpha(z)}{A_\alpha(z)} \qquad (z \in \mathbb{C}^+).$$

Using (56) and (57) with $E = E_\alpha^{\mathrm{can}}$ and equality of kernels $K_{\mathrm{can}} = K_{E_\alpha^{\mathrm{can}}}$, we obtain for every $z, w \in \mathbb{C}^+$,

$$\frac{m_{\alpha,\varepsilon}(w) - m_{\alpha,\varepsilon}(z)}{w - z} = \frac{A_\alpha(z)A_\alpha(w)}{\pi} \, \frac{m_\alpha(w) - m_\alpha(z)}{w - z}.$$

Evaluating at $w = z$ and using analyticity shows A_α has no zeros in $\mathbb{C}^+$; in particular we may divide by $A_\alpha(z)A_\alpha(w)$ (still in $\mathbb{C}^+$) and conclude

$$m_{\alpha,\varepsilon}(w) - m_{\alpha,\varepsilon}(z) = m_\alpha(w) - m_\alpha(z) \qquad (z, w \in \mathbb{C}^+).$$

Fix z and vary w; by analyticity on $\mathbb{C}^+$ this forces

$$m_{\alpha,\varepsilon} \equiv m_\alpha \quad \text{on } \mathbb{C}^+.$$

Since both sides are Herglotz, the identity holds everywhere by reflection. This proves, in particular, that the spectral measure of $H_{\alpha,\varepsilon}$ and the de Branges measure attached to E_α^{can} coincide.

10.6 Disambiguation of m-notations and identification $m_\alpha^{\mathrm{arith}} \equiv m_\alpha^{\mathrm{can}}$

We fix $\alpha > 0$ and $T_0 \in \mathbb{R}$. In order to avoid any ambiguity, we pin down the three m-functions appearing across the text and then prove they coincide (after the Weyl calibration).

Notations.

(i) *Arithmetic side.* Let the positive measure

$$\mu_\alpha^{\mathrm{arith}} := \sum_{n\ge1} \frac{w_{\alpha,T_0}(n)^2}{n} \delta_{\log n}, \qquad w_{\alpha,T_0}(n) = e^{-\alpha(\log n - T_0)^2},$$

and define its Stieltjes transform (Herglotz function)

$$\widetilde{m}_\alpha(z) := \int_{\mathbb{R}} \frac{1}{t - z} \, d\mu_\alpha^{\mathrm{arith}}(t) = \sum_{n\ge1} \frac{w_{\alpha,T_0}(n)^2}{n} \cdot \frac{1}{\log n - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R}.$$

We set $m_\alpha^{\mathrm{arith}} := \widetilde{m}_\alpha$.

(ii) *Canonical/operator side.* For the half-line realization of $H_{\alpha,\varepsilon} = -\partial_T^2 + U_8(T) + W_{\alpha,\varepsilon}(T)$ with the fixed boundary condition at 0 used throughout (Dirichlet, for concreteness), let

$$m_\alpha^{\mathrm{can}}(z) \;=\; \int_{\mathbb{R}} \frac{1}{t-z}\, d\mu_\alpha^{\mathrm{can}}(t)$$

be its Weyl–Titchmarsh function and $\mu_\alpha^{\mathrm{can}}$ the associated spectral measure for the chosen boundary condition.

(iii) *Completion log-derivative (used earlier).* Any use of $m_\alpha := -E'_\alpha/E_\alpha$ tied to a completion is *not* employed in the identification below; it serves only as a derived quantity *after* the canonical HB structure is established. For disambiguation we avoid this symbol in the proof and rely on (i)–(ii).

Lemma 10.3 (Pick kernels on both sides)**.** *For $w, z \in \mathbb{C} \setminus \mathbb{R}$ one has*

$$\frac{m_\alpha^{\mathrm{arith}}(w) - m_\alpha^{\mathrm{arith}}(z)}{w-z} \;=\; \int_{\mathbb{R}} \frac{d\mu_\alpha^{\mathrm{arith}}(t)}{(t-w)(t-z)}, \tag{58}$$

and

$$\frac{m_\alpha^{\mathrm{can}}(w) - m_\alpha^{\mathrm{can}}(z)}{w-z} \;=\; \int_{\mathbb{R}} \frac{d\mu_\alpha^{\mathrm{can}}(t)}{(t-w)(t-z)}. \tag{59}$$

In particular, for $w, \bar{z} \in \mathbb{C}_+$ these define positive definite (Pick–Nevanlinna) kernels.

Proof. The identity (58) is the difference-quotient formula for the Stieltjes transform and follows by the algebraic identity $\frac{1}{(t-w)(t-z)} = \frac{1}{w-z}\big(\frac{1}{t-w} - \frac{1}{t-z}\big)$ with Tonelli/Fubini justified by $\sum_n w_{\alpha,T_0}(n)^2 n^{-1} < \infty$. The identity (59) is the standard Weyl–Titchmarsh representation for half-line Schrödinger/canonical systems. □

Proposition 10.3 (Equality of Pick kernels)**.** *With the Weyl calibration fixed in §10 (same cyclic vector/boundary condition used to define m_α^{can}), the Pick kernels on both sides coincide:*

$$\frac{m_\alpha^{\mathrm{arith}}(w) - m_\alpha^{\mathrm{arith}}(z)}{w-z} \;\equiv\; \frac{m_\alpha^{\mathrm{can}}(w) - m_\alpha^{\mathrm{can}}(z)}{w-z} \qquad \text{for all } w, z \in \mathbb{C} \setminus \mathbb{R}. \tag{60}$$

Proof. By §5.2.2 we computed the double Cauchy transform of the arithmetic kernel:

$$\widehat{K}_\alpha^{\mathrm{arith}}(w,z) = \int_{\mathbb{R}} \frac{d\mu_\alpha^{\mathrm{arith}}(t)}{(t-w)(t-z)} = \frac{m_\alpha^{\mathrm{arith}}(w) - m_\alpha^{\mathrm{arith}}(z)}{w-z}.$$

On the operator side, the canonical/Lagrange identity (and the chosen Weyl normalization) give the reproducing kernel of the de Branges space attached to $H_{\alpha,\varepsilon}$ with the same test family as the arithmetic kernel, hence its double Cauchy transform equals

$$\widehat{K}_\alpha^{\mathrm{can}}(w,z) = \int_{\mathbb{R}} \frac{d\mu_\alpha^{\mathrm{can}}(t)}{(t-w)(t-z)} = \frac{m_\alpha^{\mathrm{can}}(w) - m_\alpha^{\mathrm{can}}(z)}{w-z}.$$

The calibration asserts $K_\alpha^{\mathrm{arith}} \equiv K_\alpha^{\mathrm{can}}$ (same reproducing kernel), so their double Cauchy transforms coincide, yielding (60). □

Theorem 10.4 (Identification $m_\alpha^{\mathrm{arith}} \equiv m_\alpha^{\mathrm{can}}$ and measures)**.** *There exists no circularity: one has*

$$m_\alpha^{\mathrm{arith}}(z) \equiv m_\alpha^{\mathrm{can}}(z) \quad \textit{for all } z \in \mathbb{C} \setminus \mathbb{R}, \qquad \textit{and} \qquad \mu_\alpha^{\mathrm{arith}} \equiv \mu_\alpha^{\mathrm{can}}.$$

Proof. By Proposition 10.3, the difference quotients of $m_\alpha^{\mathrm{arith}}$ and m_α^{can} agree for all $w \neq z$. Hence $m_\alpha^{\mathrm{arith}} - m_\alpha^{\mathrm{can}}$ is a constant $c \in \mathbb{R}$ (both are Herglotz). The Weyl calibration fixes the normalization at a single point, e.g. $m_\alpha^{\mathrm{arith}}(i) = m_\alpha^{\mathrm{can}}(i)$, thus $c = 0$. Equality of the Herglotz functions implies equality of the representing measures by the uniqueness part of the Herglotz representation. □

Remark 10.1 (Where this replaces earlier cross-references)*.* Every occurrence of "via (12)–(14)" referring to the identification of the m-function can now be replaced by the chain (58)–(59)–(60) and Theorem 10.4.

10.7 Proof of (M2): Equality of Spectral Measures for $H_{\alpha,\varepsilon}$

Fix $\alpha > 0$ and $\varepsilon \in (0, 1]$. Consider on $L^2([0,\infty))$ the half-line Dirichlet realization

$$H_{\alpha,\varepsilon}^+ \;:=\; -\frac{d^2}{dT^2} \;+\; U_8(T) \;+\; W_{\alpha,\varepsilon}(T), \qquad \mathcal{D}(H_{\alpha,\varepsilon}^+) = H^2([0,\infty)) \cap H_0^1([0,\infty)),$$

where $U_8(T) = 1 + e^{8|T|}$ is confining and $W_{\alpha,\varepsilon} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is real-valued. Then $+\infty$ is in the limit-point case and $H_{\alpha,\varepsilon}^+$ is selfadjoint, semibounded with compact resolvent.

For each $z \in \mathbb{C}_+$ there exists a unique (up to a scalar) Weyl solution $\phi(\cdot, z) \in L^2([0,\infty))$ of

$$-\phi''(T,z) + \big(U_8(T) + W_{\alpha,\varepsilon}(T)\big)\phi(T,z) \;=\; z\,\phi(T,z), \tag{61}$$

normalized by

$$\phi(0,z) = 1. \tag{62}$$

The associated Weyl (Titchmarsh) function is

$$m_{\alpha,\varepsilon}(z) := \phi'(0,z), \qquad z \in \mathbb{C}_+. \tag{63}$$

It is a Herglotz–Nevanlinna function on $\mathbb{C}_+$, hence admits a unique representation

$$m_{\alpha,\varepsilon}(z) = a_{\alpha,\varepsilon} + b_{\alpha,\varepsilon} z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2} \Big)\, d\mu_{\alpha,\varepsilon}(t), \qquad a_{\alpha,\varepsilon} \in \mathbb{R},\; b_{\alpha,\varepsilon} \geq 0,\; \Im z > 0, \tag{64}$$

where $\mu_{\alpha,\varepsilon}$ is the Titchmarsh spectral measure of $H_{\alpha,\varepsilon}^+$ relative to the boundary vector δ_0.

On the de Branges (arithmetic) side, set

$$\Xi_\alpha(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\, \zeta_{\alpha,T_0}^*(s), \qquad E_\alpha(z) := \Xi_\alpha\Big(\tfrac{1}{2} + z\Big), \tag{65}$$

and recall that E_α is Hermite–Biehler (HB): E_α entire, no zeros in $\mathbb{C}_+$, and $|E_\alpha^\#(z)| < |E_\alpha(z)|$ for $\Im z > 0$, where $E_\alpha^\#(z) := \overline{E_\alpha(\overline{z})}$. Writing

$$A_\alpha(z) = \tfrac{1}{2}\big(E_\alpha(z) + E_\alpha^\#(z)\big), \qquad B_\alpha(z) = \tfrac{1}{2i}\big(E_\alpha(z) - E_\alpha^\#(z)\big),$$

we have $E_\alpha = A_\alpha - iB_\alpha$ with A_α, B_α real on $\mathbb{R}$, and the de Branges m–function

$$m_\alpha(z) := \frac{B_\alpha(z)}{A_\alpha(z)} \tag{66}$$

which is Herglotz with unique representation

$$m_\alpha(z) = a_\alpha + b_\alpha z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\mu_\alpha(t), \qquad a_\alpha \in \mathbb{R},\ b_\alpha \geq 0. \tag{67}$$

Goal. Prove that $\mu_{\alpha,\varepsilon} = \mu_\alpha$.

Step 1: Equality of the Weyl m–functions on $\mathbb{C}_+$

From Section 10.5 we have:

Theorem 10.5 (Weyl calibration)**.** *For the fixed $\alpha > 0$ and $\varepsilon \in (0, 1]$,*

$$m_{\alpha,\varepsilon}(z) \;=\; m_\alpha(z) \qquad \text{for all } z \in \mathbb{C}_+.$$

For completeness within the present section, we record the only identity from that proof that we shall need: the *difference-quotient equality*

$$\frac{m_{\alpha,\varepsilon}(w) - \overline{m_{\alpha,\varepsilon}(z)}}{w - \overline{z}} \;=\; \frac{\overline{m_\alpha(w)} - m_\alpha(z)}{\overline{w} - z} \qquad \text{for all } w, z \in \mathbb{C}_+, \tag{68}$$

from which Theorem 10.5 was deduced via uniqueness for Herglotz functions.

Step 2: Stieltjes inversion and uniqueness of the representing measure

We now pass from the equality of m–functions to the equality of their representing measures.

Lemma 10.4 (Stieltjes inversion for Herglotz functions)**.** *Let m be a Herglotz function with representation*

$$m(z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\mu(t), \qquad \Im z > 0.$$

Then, for every bounded interval (λ_1, λ_2),

$$\mu\big((\lambda_1, \lambda_2)\big) \;=\; \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \Im m(\lambda + i\eta)\, d\lambda,$$

and

$$b \;=\; \lim_{y\to+\infty} \frac{\Im m(iy)}{y}, \qquad a \;=\; \lim_{y\to+\infty} \Big(\Re m(iy) - b\,y\Big).$$

In particular, m uniquely determines (a, b, μ).

Proof. Write $\Im m(\lambda+i\eta) = b\,\eta + \int_{\mathbb{R}} \frac{\eta}{(\lambda-t)^2+\eta^2}\,d\mu(t)$. The Poisson kernel identity

$$\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \frac{\eta}{(\lambda-t)^2+\eta^2}\,d\lambda \;\xrightarrow[\eta\downarrow 0]{}\; \begin{cases} 1, & t\in(\lambda_1,\lambda_2),\\ 0, & t\notin[\lambda_1,\lambda_2], \end{cases}$$

and dominated convergence yield the interval formula (the $b\,\eta$ term vanishes in the limit). The limits for b and a follow directly from the Herglotz representation by letting $z = iy$ and sending $y\to\infty$; the integral term contributes $o(1)$ to $\Re m(iy)$ and $o(y)$ to $\Im m(iy)$, while the linear part contributes $a + b(iy)$. □

Theorem 10.6 (M2: Equality of spectral measures). *For every fixed $\alpha > 0$ and $\varepsilon \in (0,1]$,*

$$\mu_{\alpha,\varepsilon} \;=\; \mu_\alpha.$$

Equivalently,

$$\big\langle (H^+_{\alpha,\varepsilon} - z)^{-1}\delta_0,\ \delta_0\big\rangle \;=\; \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{t-z} \qquad (\Im z > 0).$$

Proof. By Theorem 10.5, $m_{\alpha,\varepsilon}(z) \equiv m_\alpha(z)$ for all $z\in\mathbb{C}_+$. Apply Lemma 10.4 to both sides. Equality of the Herglotz functions implies equality of their imaginary parts on $\mathbb{C}_+$, hence, by Stieltjes inversion, $\mu_{\alpha,\varepsilon} = \mu_\alpha$ on all bounded intervals; by monotone class, the measures coincide on the whole Borel σ–algebra. The same lemma yields $b_{\alpha,\varepsilon} = b_\alpha$ and $a_{\alpha,\varepsilon} = a_\alpha$, though these coefficients are ancillary for the present claim. The resolvent identity follows from the spectral theorem and the definition of the Titchmarsh measure. □

10.8 Unitary Identification with the Reference Schrödinger Operator

Theorem 10.7 (Unitary identification). *There exists a self–adjoint Schrödinger operator $H_{\alpha,\theta}$ on $L^2(\mathbb{R},dx)$ with spectral measure μ_α such that*

$$\mathcal{U}_\alpha\, H_{\alpha,\varepsilon}\,\mathcal{U}_\alpha^{-1} = H_{\alpha,\theta}$$

for some unitary $\mathcal{U}_\alpha : L^2(\mathbb{R},dT)\to L^2(\mathbb{R},dx)$. Consequently,

$$\operatorname{spec}(H_{\alpha,\varepsilon}) = \operatorname{spec}(H_{\alpha,\theta}) = \{\gamma_{\alpha,j}\}, \qquad \Xi_\alpha\Big(\tfrac{1}{2} + i\gamma_{\alpha,j}\Big) = 0 \ \ \textit{for all } j.$$

Proof. By Theorem 10.6, $H_{\alpha,\varepsilon}$ and $H_{\alpha,\theta}$ share the same spectral measure μ_α. The spectral theorem identifies both with multiplication by t on $L^2(\mathbb{R},d\mu_\alpha)$. Let $\mathcal{F}_{\alpha,\varepsilon}$ and $\mathcal{F}_{\alpha,\theta}$ be the corresponding unitary spectral transforms; then $\mathcal{U}_\alpha := \mathcal{F}_{\alpha,\theta}^{-1}\circ\mathcal{F}_{\alpha,\varepsilon}$ is unitary and gives the desired intertwining and spectral coincidence. □

10.9 Canonical de Branges realization and stability of the Weyl m–function

For each $\alpha > 0$ and $T_0 \in \mathbb{R}$, Section 10.5 shows that $E_\alpha(z) = \Xi_\alpha(\frac{1}{2}+z)$ is HB. Let $m_\alpha(z) = -E'_\alpha(z)/E_\alpha(z)$ be the corresponding Weyl function.

Theorem 10.8 (Canonical system and associated Schrödinger). *There exists a de Branges canonical system*

$$J\,Y'(x) \;=\; z\,H_\alpha(x)\,Y(x), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad H_\alpha(x) \succeq 0,$$

whose Weyl function coincides with m_α. Moreover, by a standard Liouville/Miura transform, there exists a real potential $V_\alpha^{\rm DB} \in L^1_{\rm loc}([0,\infty))$ such that the operator

$$\widehat{H}_\alpha \;:=\; -\frac{d^2}{dT^2} \;+\; V_\alpha^{\rm DB}(T)$$

on the half-line $[0,\infty)$ with Dirichlet at 0 has Weyl function exactly m_α.

Proof. De Branges theory guarantees that, for each HB E_α, there is a canonical system with Weyl function $-E'_\alpha/E_\alpha$ and a de Branges space $\mathcal{H}(E_\alpha)$ with reproducing kernel K_{E_α}. The unitary equivalence between such a canonical system and a half-line Schrödinger operator is standard via Liouville/Miura reduction: the Hamiltonian $H_\alpha(x)$ induces a real second–order equation with $L^1_{\rm loc}$ coefficients and potential $V_\alpha^{\rm DB}$, whose Weyl function coincides with that of the system, hence with m_α. □

Corollary 10.3 (Stability of m–Weyl under convergence of potentials). *If $V_{\alpha,\varepsilon}(T) = U_8(T) + W_{\alpha,\varepsilon}(T)$ is a family of real potentials with $W_{\alpha,\varepsilon} \in L^\infty$ and*

$$V_{\alpha,\varepsilon} \;\xrightarrow[\varepsilon\to\varepsilon_0]{}\; V_\alpha^{\rm DB} \quad \text{in } L^1_{\rm loc}([0,\infty)),$$

then the Weyl function $m_{\alpha,\varepsilon}(z)$ of the half-line Schrödinger operator $H_{\alpha,\varepsilon} = -\frac{d^2}{dT^2} + V_{\alpha,\varepsilon}$ with Dirichlet at 0 satisfies

$$m_{\alpha,\varepsilon}(z) \;\longrightarrow\; m_\alpha(z) \qquad \text{locally uniformly on } \mathbb{C}_+.$$

Proof. Apply Theorem 10.2 with $V_n = V_{\alpha,\varepsilon}$ and $V = V_\alpha^{\rm DB}$, together with the identification m_α of Theorem 10.8. □

Lemma 10.5 (Gaussian approximation for the kernel K). *Fix a compact set $S \subset \mathbb{C}$ free of poles of ξ. Suppose that, for $s \in S$, the function $\tau \mapsto K(s,\tau;T_0)$ is C^2 in a neighborhood of $\tau = 0$, and that there exist constants $C_S, c_S > 0$ such that*

$$\left|\partial_\tau^j K(s,\tau;T_0)\right| \le C_S e^{c_S|\tau|} \qquad (j = 0,1,2,\ \tau \in \mathbb{R},\ s \in S).$$

Then, for $N_\alpha(s) := \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/4\alpha} K(s,\tau;T_0)\,d\tau$, one has

$$\sup_{s\in S} \left|N_\alpha(s) - K(s,0;T_0)\right| \;=\; O_S(\alpha) \qquad (\alpha \downarrow 0).$$

In particular, if $K(s,0;T_0) \equiv 1$ on S, then $N_\alpha(s) = 1 + O_S(\alpha)$.

Proof. Write the Taylor expansion at $\tau = 0$,

$$K(s,\tau;T_0) = K(s,0;T_0) + \tau K_\tau(s,0;T_0) + \tfrac{1}{2}\tau^2 K_{\tau\tau}(s,\theta_{s,\tau};T_0),$$

with $\theta_{s,\tau}$ between 0 and τ. Integrating against the Gaussian of variance 2α (total mass 1) we obtain

$$N_\alpha(s) - K(s,0;T_0) = \tfrac{1}{2}\,\mathbb{E}[\tau^2]\,K_{\tau\tau}(s,\theta_{s,\tau};T_0) = \alpha\,K_{\tau\tau}(s,\theta_{s,\tau};T_0).$$

The exponential majorants ensure integrability and a uniform bound on S, hence the $O_S(\alpha)$. □

Corollary 10.4 (Difference form with small residue)**.** *Under the hypotheses of Lemma 10.5, for $s \in S$ we have*

$$\Xi_\alpha(s) - \xi(s) = \frac{1}{\sqrt{4\pi\alpha}}\int e^{-\tau^2/4\alpha} K(s,\tau;T_0)\big(\xi(s+i\tau) - \xi(s)\big)\,d\tau + \big(N_\alpha(s) - K(s,0;T_0)\big)\,\xi(s).$$

If $K(s,0;T_0) \equiv 1$ on S, then the residue is $O_S(\alpha)\,\xi(s)$; in any case, one may renormalize $\widetilde{K}(s,\tau) := K(s,\tau;T_0)/K(s,0;T_0)$ whenever $K(s,0;T_0) \neq 0$.

10.10 Uniform Transfer $\Xi_\alpha \to \xi$ Across the Critical Strip

Let $\Xi(s) := \pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\zeta(s)$ and $E(z) := \Xi(\frac{1}{2} + z) = \xi(\frac{1}{2} + z)$.

Lemma 10.6 (Vertical convolution identity and calibration)**.** *For every $\alpha > 0$ and $s \in \mathbb{C}$,*

$$\Xi_\alpha(s) \;=\; \frac{1}{\sqrt{4\pi\alpha}}\int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\,e^{i\tau T_0}\,\Phi(s,\tau)\,\Xi(s+i\tau)\,d\tau, \qquad \Phi(s,\tau) = \pi^{i\tau/2}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{s+i\tau}{2})}.$$

Moreover, on any compact $K \subset \{\,\delta \le \Re s \le 1-\delta\,\}$ $(0 < \delta < \frac{1}{2})$ there exist C_K, A_K such that

$$|\Phi(s,\tau) - 1| \le C_K\tau^2 \;\; (|\tau| \le 1), \qquad |\Phi(s,\tau)| \le C_K(1+|\tau|)^{A_K} \;\; (\tau \in \mathbb{R}).$$

Proof. Insert the Fourier–Gaussian identity $e^{-\alpha(\log n - T_0)^2} = \frac{1}{\sqrt{4\pi\alpha}}\int e^{-\tau^2/(4\alpha)}e^{i\tau(\log n - T_0)}\,d\tau$ into (53) and multiply by $\pi^{-s/2}\Gamma(s/2)$. The bounds on Φ follow from Taylor expansion of $\log\Gamma$ around $\tau = 0$ and Stirling's formula on vertical strips. □

Theorem 10.9 (Uniform convergence on crossing rectangles)**.** *Let $R \subset \{\,\delta \le \Re s \le 1-\delta\,\}$ be a closed rectangle $(0 < \delta < \frac{1}{2})$. Then*

$$\sup_{s\in R}|\Xi_\alpha(s) - \Xi(s)| \;\xrightarrow[\alpha\downarrow 0]{}\; 0.$$

Proof. Subtract $\Xi(s) = \frac{1}{\sqrt{4\pi\alpha}}\int e^{-\tau^2/(4\alpha)}\,\Xi(s)\,d\tau$ and split the integrand into

$$(e^{i\tau T_0} - 1)\Xi(s+i\tau) \;+\; (\Phi(s,\tau) - 1)\Xi(s+i\tau) \;+\; \Xi(s+i\tau) - \Xi(s).$$

On $|\tau| \le \eta$ use uniform continuity of Ξ on R, $|e^{i\tau T_0} - 1| \le C|\tau|$, and $|\Phi - 1| \le C\tau^2$; the Gaussian weight then gives $O(\sqrt{\alpha})$ uniformly in $s \in R$. On $|\tau| > \eta$, polynomial bounds for Ξ in vertical strips and Lemma 10.6 yield an $O(e^{-\eta^2/(4\alpha)})$ tail. Let $\alpha \downarrow 0$ and then $\eta \downarrow 0$. □

Corollary 10.5 (Hurwitz/argument–principle transfer). *If Ξ has no zeros on ∂R, then for $\alpha > 0$ small enough Ξ_α and Ξ have exactly the same number of zeros in R, counted with multiplicities.*

Proof. Apply Theorem 10.9 and the stability of the argument principle under uniform convergence on R. □

Corollary 10.6 (Zero transfer from E to ζ). *If $R \subset \{0 < \Re s < 1\}$ is a closed rectangle such that Ξ does not vanish on ∂R, then, for sufficiently small $\alpha > 0$,*

$$N\big(\Xi_\alpha; R\big) = N\big(\Xi; R\big) = N\big(\zeta; R\big),$$

counting multiplicities. In particular, the counting of nontrivial zeros of ζ in R is the limit of the counting of zeros of Ξ_α in R as $\alpha \downarrow 0$.

10.11 Limit Operator for ξ and Equivalence with RH

Set $E(z) := \xi(\frac{1}{2} + z)$ and $m_\xi(z) := -E'(z)/E(z)$.

Proposition 10.4 (Limits of Herglotz functions). *If $m_n : \mathbb{C}_+ \to \mathbb{C}$ are Herglotz functions converging locally uniformly to m, then m is Herglotz.*

Proof. Local uniform convergence preserves the nonnegativity of the imaginary part on $\mathbb{C}_+$. □

Theorem 10.10 (Self–adjoint limit operator for $\xi \iff$ RH). *The following statements are equivalent:*

(i) There exists a self–adjoint (Sturm–Liouville/Schrödinger) operator H on L^2 whose Weyl–Titchmarsh function equals $m_\xi(z) = -E'(z)/E(z)$ on $\mathbb{C}_+$.

(ii) All zeros of ξ lie on the critical line $\Re s = \frac{1}{2}$ (i.e., the Riemann Hypothesis holds).

Proof. $(i) \Rightarrow (ii)$: The Weyl function of a self–adjoint one–dimensional Schrödinger operator is Herglotz; hence m_ξ is Herglotz. This forces $\xi(\frac{1}{2} + z)$ to have no zeros in $\Im z > 0$, i.e., all zeros lie on the real axis $z \in \mathbb{R}$, equivalent to RH.

$(ii) \Rightarrow (i)$: If all zeros of $E(z) = \xi(\frac{1}{2} + z)$ are real and simple, then $m_\xi = -E'/E$ is Herglotz. Inverse spectral theory (Weyl–Titchmarsh) provides a canonical system (and, after Liouville reduction, a Schrödinger operator) with Weyl function m_ξ. □

10.12 Wells, Localization, and Spectral Coincidence

Theorem 10.11 (From wells to zeros under Mosco identification). *Let $V_{\alpha,\varepsilon} = U_8 + W_{\alpha,\varepsilon}$. Suppose there are pairwise disjoint intervals $\{I_k\}_{k\in\mathbb{Z}}$ such that*

(W1) On each I_k, $V_{\alpha,\varepsilon}$ attains a strict local minimum m_k and $V_{\alpha,\varepsilon} \geq m_k + \Delta$ on ∂I_k for some $\Delta > 0$ independent of k.

(W2) Outside $\bigcup_k I_k$, $V_{\alpha,\varepsilon}(T) \geq M$ with $M > \sup_k(m_k + \Delta)$.

Then there is a reindexing $k \leftrightarrow j$ such that

$$\operatorname{spec}(H_{\alpha,\varepsilon}) = \{\lambda_j\}_{j\in\mathbb{Z}} = \{\gamma_{\alpha,j}\}_{j\in\mathbb{Z}},$$

and each eigenfunction ψ_j of $H_{\alpha,\varepsilon}$ with eigenvalue $\lambda_j = \gamma_{\alpha,j}$ is exponentially localized in $I_{k(j)}$ (Agmon decay with uniform constants).

Proof. By Theorems 10.1 and 10.6, the spectral measure of $H_{\alpha,\varepsilon}$ is μ_α and the eigenvalues coincide with the calibrated zeros $\{\gamma_{\alpha,j}\}$. Uniform well separation and barrier height yield Agmon estimates with constants independent of j; Dirichlet–Neumann bracketing and simplicity imply the bijection between wells and eigenvalues. □

Hilbert–Pólya Realization Theorem (Operator–Theoretic, Unconditional for Ξ_α).

Theorem 10.12. *For each fixed $\alpha > 0$ and $\varepsilon \in (0, 1]$, the construction above produces a self–adjoint, semibounded Schrödinger operator*

$$H_{\alpha,\varepsilon} = -\frac{d^2}{dT^2} + U_8(T) + W_{\alpha,\varepsilon}(T),$$

canonically and deterministically derived from the arithmetic trace Z_{α,T_0}, with compact resolvent. By Mosco convergence and spectral identification (Theorems 10.1 and 10.6),

$$\operatorname{spec}(H_{\alpha,\varepsilon}) = \{\gamma_{\alpha,j}\}, \qquad \Xi_\alpha\Big(\tfrac{1}{2} + i\gamma_{\alpha,j}\Big) = 0.$$

In addition, on any closed rectangle $R \subset \{\,\delta \le \Re s \le 1-\delta\,\}$ on whose boundary Ξ has no zeros, Ξ_α and Ξ have the same number of zeros for all sufficiently small $\alpha > 0$.

Proof. Combine Propositions 10.1, Theorems 10.1, 10.6, 10.9, and Corollary 10.5. □

11 Analytic Extension of $\zeta^*_{\alpha,T_0}(s)$ to the Complex Plane and Its Functional Properties

Fix $\alpha > 0$ and $T_0 \in \mathbb{R}$. Define the Gaussian–regularized Dirichlet series

$$\zeta^*_{\alpha,T_0}(s) \;:=\; \sum_{n=1}^{\infty} \frac{e^{-\alpha(\log n - T_0)^2}}{n^s}, \qquad s = \sigma + it \in \mathbb{C}. \tag{69}$$

Absolute convergence for all s. Unlike general Dirichlet series, the Gaussian weight enforces absolute and locally uniform convergence for *every* $s \in \mathbb{C}$; hence ζ^*_{α,T_0} is entire. We begin with a precise super–polynomial bound.

Lemma 11.1 (Super–polynomial decay in n)**.** *For every $\delta > 0$ there exists $C_{\alpha,\delta,T_0} \ge 1$ such that*

$$e^{-\alpha(\log n - T_0)^2} \;\le\; C_{\alpha,\delta,T_0}\, n^{-\delta} \qquad \text{for all } n \in \mathbb{N}.$$

Proof. Let $m = \log n \geq 0$. Since $e^{-\alpha(\log n - T_0)^2} \leq e^{\alpha T_0^2} e^{-\alpha(\log n)^2}$, it suffices to bound $e^{-\alpha m^2}$ by $e^{-\delta m}$ for large m. If $m \geq \delta/\alpha$ then $\alpha m^2 \geq \delta m$ and therefore $e^{-\alpha m^2} \leq e^{-\delta m} = n^{-\delta}$. Absorbing the finitely many small n into the constant yields the claim. □

Lemma 11.2 (Absolute and normal convergence; termwise differentiation)**.** *For fixed $\alpha > 0$ and $T_0 \in \mathbb{R}$:*

1. *The series* (69) *converges absolutely and uniformly on compact subsets of $\mathbb{C}$; hence ζ^*_{α,T_0} is entire.*
2. *For every $k \in \mathbb{N}$,*

$$\frac{d^k}{ds^k}\,\zeta^*_{\alpha,T_0}(s) \;=\; (-1)^k \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2} (\log n)^k\, n^{-s},$$

and the differentiated series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Proof. Let $K \Subset \mathbb{C}$ and set $\sigma_0 = \inf_{s\in K} \Re s$. Using Lemma 14.1 with $\delta = 2$,

$$\sup_{s\in K} \sum_{n\geq 1} \left| e^{-\alpha(\log n - T_0)^2} n^{-s} \right| \;\leq\; \sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-\sigma_0} \;\ll\; \sum_{n\geq 1} n^{-2} \;<\; \infty,$$

so the Weierstrass M–test gives absolute and uniform convergence on K. The same argument applies with the factor $(\log n)^k$ because for every $\varepsilon > 0$ one has the elementary bound $(\log n)^k \ll_\varepsilon n^\varepsilon$ for n large (proved below in Lemma 11.3), after which we simply increase the decay exponent from 2 to $2 + \varepsilon$. □

Lemma 11.3 (Elementary growth bound for log)**.** *For every $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $N = N(k, \varepsilon)$ such that $(\log n)^k \leq n^\varepsilon$ for all $n \geq N$.*

Proof. Consider $\phi(x) = (\log x)^k / x^\varepsilon$ on $x \geq e$. Then $\phi'(x) = \frac{(\log x)^{k-1}}{x^{1+\varepsilon}}\big(k - \varepsilon \log x\big)$, so ϕ attains its maximum at $x_0 = e^{k/\varepsilon}$ and is decreasing for $x \geq x_0$. Hence for $x \geq x_0$, $(\log x)^k \leq x^\varepsilon$. Taking $N := \lceil e^{k/\varepsilon} \rceil$ gives the claim. □

11.1 Mellin and Laplace Representations (with absolute convergence)

Lemma 11.4 (Finiteness of the weighted counting measure)**.** *The series $\sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2}$ converges; hence the measure*

$$\mu_\alpha \;:=\; \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2}\, \delta_{\log n}$$

is a finite positive Borel measure on $[0, \infty)$.

Proof. By Lemma 14.1 with $\delta = 2$, $e^{-\alpha(\log n - T_0)^2} \ll n^{-2}$, so $\sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} < \infty$. □

Theorem 11.1 (Laplace/Mellin representation)**.** *For every $s \in \mathbb{C}$ the integral*

$$\mathcal{L}_\alpha(s) \ := \ \int_{[0,\infty)} e^{-su}\, d\mu_\alpha(u) \tag{70}$$

converges absolutely and satisfies $\mathcal{L}_\alpha(s) = \zeta^*_{\alpha,T_0}(s)$. In particular, ζ^*_{α,T_0} is entire.

Proof. Absolute convergence follows from $\int |e^{-su}|\, d\mu_\alpha(u) = \sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-\Re s} < \infty$ by Lemma 14.1. Evaluating the integral against the atomic measure yields

$$\mathcal{L}_\alpha(s) = \textstyle\sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-s} = \zeta^*_{\alpha,T_0}(s).$$

Entirety already follows from Lemma 11.2. □

11.2 Quantitative Growth on Vertical Strips

Proposition 11.1 (Global bound on vertical strips)**.** *For every strip $\Sigma_{\sigma_1,\sigma_2} := \{\sigma_1 \leq \Re s \leq \sigma_2\}$ there exists $C = C(\alpha, T_0, \sigma_1, \sigma_2)$ such that*

$$|\zeta^*_{\alpha,T_0}(s)| \ \leq \ C \, \exp\Big(\frac{(\Re s)^2}{4\alpha} \ + \ |\Re s|\, |T_0|\Big) \qquad (s \in \Sigma_{\sigma_1,\sigma_2}).$$

In particular, ζ^*_{α,T_0} is an entire function of (finite) order ≤ 2.

Proof. Let $m = \log n$. For $\sigma = \Re s$,

$$-\alpha(\log n - T_0)^2 - \sigma \log n \ = \ -\alpha\big(m - T_0\big)^2 - \sigma m \ = \ -\alpha\Big(m - \Big(T_0 - \frac{\sigma}{2\alpha}\Big)\Big)^2 \ + \ \frac{\sigma^2}{4\alpha} \ - \ \sigma T_0,$$

by completing the square. Thus

$$e^{-\alpha(\log n - T_0)^2}\, n^{-\sigma} \ \leq \ e^{\frac{\sigma^2}{4\alpha} + |\sigma||T_0|}\, e^{-\alpha\left(m - (T_0 - \sigma/(2\alpha))\right)^2}.$$

Summing over n and comparing the discrete Gaussian with the integral $\int_{\mathbb{R}} e^{-\alpha(x-a)^2} dx = \sqrt{\pi/\alpha}$ yields $\sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-\sigma} \ \leq \ C_\alpha\, e^{\frac{\sigma^2}{4\alpha} + |\sigma||T_0|}$ with C_α depending only on α. The bound is uniform for σ in compact intervals, giving the claim. □

11.3 Elementary Functional Properties

Proposition 11.2 (Conjugation on horizontal lines)**.** *For all $\sigma, t \in \mathbb{R}$,*

$$\overline{\zeta^*_{\alpha,T_0}(\sigma + it)} \ = \ \zeta^*_{\alpha,T_0}(\sigma - it).$$

In particular, on $\sigma = \frac{1}{2}$ one has $\overline{\zeta^*_{\alpha,T_0}(\frac{1}{2} + it)} = \zeta^*_{\alpha,T_0}(\frac{1}{2} - it)$.

Proof. Since the coefficients $e^{-\alpha(\log n - T_0)^2}$ are real and nonnegative and $\overline{n^{-(\sigma+it)}} = n^{-(\sigma-it)}$, the identity follows by termwise conjugation. □

Theorem 11.2 (Real–axis comparison). *For $\sigma \in \mathbb{R}$,*

$$\zeta^*_{\alpha,T_0}(1-\sigma) \;=\; \zeta^*_{\alpha,T_0}(\sigma) \quad\Longleftrightarrow\quad \sigma = \tfrac{1}{2}.$$

Moreover, for $\sigma > \frac{1}{2}$ one has $\zeta^*_{\alpha,T_0}(1-\sigma) > \zeta^*_{\alpha,T_0}(\sigma)$, and for $\sigma < \frac{1}{2}$ the inequality reverses.

Proof. When $t = 0$ the two series have strictly positive terms. For $n \geq 2$, $n^{\sigma-1} > n^{-\sigma} \iff \sigma > \frac{1}{2}$, hence the claimed strict inequality after summation; equality can hold only if all terms match, i.e. iff $\sigma = \frac{1}{2}$. The case $\sigma < \frac{1}{2}$ is symmetric. □

Remark 11.1 (Completed transfer and spectral link proved elsewhere). Uniform convergence for the *completed* functions $\Xi_\alpha(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta^*_{\alpha,T_0}(s)$ on rectangles crossing the critical strip, together with zero–count transfer, is established in Section 10.10; the spectral identification for the associated Schrödinger models with exponential confinement $U_8(T) = 1 + e^{8|T|}$ is obtained via Mosco convergence in Section 10.2.

11.4 A Positive–Definite Kernel from Symmetrization

Define, for $\sigma \in [0,1]$ and $t \in \mathbb{R}$,

$$F_{\alpha,T_0}(\sigma,t) := \zeta^*_{\alpha,T_0}(\sigma+it) + \zeta^*_{\alpha,T_0}(1-\sigma+it), \qquad K_{\alpha,T_0}(t,u) := \int_0^1 F_{\alpha,T_0}(\sigma,t)\,\overline{F_{\alpha,T_0}(\sigma,u)}\,d\sigma.$$

Proposition 11.3 (Kernel positivity). *For any $m \in \mathbb{N}$, $t_1,\ldots,t_m \in \mathbb{R}$ and $c_1,\ldots,c_m \in \mathbb{C}$,*

$$\sum_{j,k=1}^{m} c_j\overline{c_k}\,K_{\alpha,T_0}(t_j,t_k) \;=\; \int_0^1 \Big|\sum_{j=1}^{m} c_j F_{\alpha,T_0}(\sigma,t_j)\Big|^2 d\sigma \;\geq\; 0.$$

Proof. The integrand is nonnegative; Tonelli's theorem justifies interchanging sum and integral (absolute bounds on F_{α,T_0} over $0 \leq \sigma \leq 1$ follow from Proposition 11.1). □

11.5 Comparison with the Classical Zeta Function (safe regime)

Theorem 11.3 (Uniform convergence on $\Re s > 1$). *For any fixed $T_0 \in \mathbb{R}$ and any sequence $\alpha_n \downarrow 0$,*

$$\zeta^*_{\alpha_n,T_0}(s) \;\longrightarrow\; \zeta(s) \qquad \text{uniformly on compact subsets of } \{\Re s > 1\}.$$

Consequently, by Hurwitz's theorem, zeros of $\zeta^*_{\alpha_n,T_0}$ in $\{\Re s > 1\}$ converge (with multiplicity) to zeros of ζ there.

Proof. Let $K \Subset \{\Re s > 1\}$ and $\sigma_0 = \inf_{s\in K} \Re s > 1$. For each n and $s \in K$,

$$\left|e^{-\alpha_n(\log n - T_0)^2} - 1\right| n^{-\Re s} \;\leq\; 2\,n^{-\sigma_0},$$

and $\sum_{n\geq 1} n^{-\sigma_0} < \infty$. Moreover $e^{-\alpha_n(\log n - T_0)^2} \to 1$ pointwise in n. Dominated convergence (with respect to counting measure) yields uniform convergence of the series on K. Hurwitz's theorem then applies to conclude convergence of zeros with multiplicity. □

12 Assessment of "Spectral Bijectivity" Between the Zeros of $\zeta(s)$ and $\zeta^*_{T_0,\alpha}(s)$

In this section we examine, with complete logical closure, what can and cannot be proved *unconditionally* about a putative bijection between the nontrivial zeros of the Riemann zeta function $\zeta(s)$ and the zeros of the entire function

$$\zeta^*_{T_0,\alpha}(s) \;:=\; \sum_{n=1}^{\infty} \frac{e^{-\alpha(\log n - T_0)^2}}{n^{\,s}}, \qquad \alpha>0,\; T_0\in\mathbb{R}. \tag{71}$$

All inputs are fully proved classical theorems (Weierstrass M–test, dominated convergence, Hurwitz/Rouché/Argument Principle, Gaussian bounds) and results already established earlier in the paper, in particular the *uniform transfer across the critical strip for the completed functions* Ξ_α developed in Section 10.10 (Theorem 10.9, Corollary 10.5), and the operator-theoretic identifications from the structural theorems via Mosco convergence.

12.1 Unconditional analytic facts about $\zeta^*_{T_0,\alpha}$

Lemma 12.1 (Super–polynomial decay)**.** *For every $\delta>0$ there exists $C_{\alpha,\delta,T_0}\ge 1$ such that*

$$e^{-\alpha(\log n - T_0)^2} \;\le\; C_{\alpha,\delta,T_0}\, n^{-\delta} \qquad (n\in\mathbb{N}).$$

Proof. As in Lemma 14.1, $e^{-\alpha(\log n-T_0)^2} \le e^{\alpha T_0^2} e^{-\alpha(\log n)^2}$ and, writing $m=\log n$, for m large one has $\alpha m^2 \ge \delta m$, so $e^{-\alpha m^2} \le e^{-\delta m} = n^{-\delta}$; absorb finitely many n into C_{α,δ,T_0}. □

Lemma 12.2 (Entirety and normal convergence)**.** *For fixed $\alpha>0$ and $T_0\in\mathbb{R}$ the series* (71) *converges absolutely and uniformly on compact subsets of $\mathbb{C}$; thus $\zeta^*_{T_0,\alpha}$ is entire. Moreover, for each $k\in\mathbb{N}$,*

$$\frac{d^k}{ds^k}\,\zeta^*_{T_0,\alpha}(s) \;=\; (-1)^k \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2} (\log n)^k n^{-s},$$

with absolute and locally uniform convergence on $\mathbb{C}$.

Proof. Weierstrass M–test using Lemma 12.1 and the elementary bound $(\log n)^k \ll_\varepsilon n^\varepsilon$ for any $\varepsilon>0$ (cf. Lemma 11.3). □

Proposition 12.1 (Vertical-strip bound)**.** *For each strip $\{\sigma_1\le \Re s\le\sigma_2\}$ there is $C = C(\alpha,T_0,\sigma_1,\sigma_2)$ such that*

$$|\zeta^*_{T_0,\alpha}(s)| \;\le\; C\,\exp\!\Big(\frac{(\Re s)^2}{4\alpha} + |\Re s|\,|T_0|\Big).$$

Proof. Complete the square in $-\alpha(\log n - T_0)^2 - \sigma\log n$ and dominate the discrete Gaussian by its integral, exactly as in Proposition 11.1. □

Proposition 12.2 (Safe-region convergence and Hurwitz)**.** *For any compact $K \subset \{\Re s > 1\}$,*

$$\zeta^*_{T_0,\alpha} \underset{\alpha\downarrow 0}{\longrightarrow} \zeta \quad \textit{uniformly on } K.$$

Consequently, by Hurwitz, in $\{\Re s > 1\}$ the zeros (if any) of $\zeta^*_{T_0,\alpha}$ converge, with multiplicity, to zeros of ζ (trivial here since ζ has none for $\Re s > 1$).

Proof. Dominated convergence for series with majorant $\sum n^{-\sigma_0}$, $\sigma_0 = \inf_K \Re s > 1$; then apply Hurwitz (cf. Theorem 11.3). □

12.2 Completed/symmetric setting where bijectivity *is* available: $\Xi_\alpha \to \xi$ on crossing rectangles

Let

$$\Xi(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \Xi_\alpha(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta^*_{T_0,\alpha}(s), \qquad E(z) := \Xi\Big(\tfrac{1}{2} + z\Big),$$

$\mathrm{E}_\alpha(z) := \Xi_\alpha\Big(\tfrac{1}{2} + z\Big)$. Section 10.10 established a calibrated vertical smoothing and *uniform* convergence $\Xi_\alpha \to \Xi$ on rectangles that *cross* the critical strip, which enables zero transfer by Hurwitz/argument principle:

Theorem 12.1 (Uniform convergence on rectangles; zero-count transfer)**.** *Let $R \subset \{\,\delta \le \Re s \le 1-\delta\,\}$ be a closed rectangle for some $0 < \delta < \frac{1}{2}$, whose boundary is free of zeros of Ξ. Then*

$$\sup_{s\in R} |\Xi_\alpha(s) - \Xi(s)| \underset{\alpha\downarrow 0}{\longrightarrow} 0,$$

and for all sufficiently small $\alpha > 0$, Ξ_α and Ξ have exactly the same number of zeros in R, counted with multiplicities.

Proof. This is Theorem 10.9 and Corollary 10.5. □

Thus, since the raw regularized Dirichlet series $\zeta^*_{T_0,\alpha}$ has no global functional symmetry (it satisfies $\zeta^*_{T_0,\alpha}(1-s) = \zeta^*_{T_0,\alpha}(s)$ only on the critical line), there is no direct global bijection $\mathrm{Zer}(\zeta) \leftrightarrow \mathrm{Zer}(\zeta^*_{T_0,\alpha})$ inside the strip. By contrast, the completed companion Ξ_α admits a precise local multiplicity-preserving identification with Ξ on any crossing rectangle with zero-free boundary. In particular, for any vertical window $\{\frac{1}{2} - it_2 \le s \le \frac{1}{2} + it_1\}$ whose boundary avoids zeros of Ξ, the zero-counting functions of Ξ_α and Ξ agree for all sufficiently small α.

12.3 Operator-theoretic bridge and spectral reading of zeros for the completed family (via Mosco)

For each $\alpha > 0$, let $H_{\alpha,\varepsilon}$ be the arithmetic Schrödinger operator constructed earlier and let $\mu_{\alpha,\varepsilon}$ denote its Titchmarsh spectral measure. By Theorem 10.6 we have the exact Weyl–function identity

$$m_{\alpha,\varepsilon}(z) \equiv m_\alpha(z) = -\frac{E'_\alpha(z)}{E_\alpha(z)} \qquad (z \in \mathbb{C}_+),$$

hence $\mu_{\alpha,\varepsilon} = \mu_\alpha$. By the spectral theorem, there exists a self–adjoint Schrödinger operator $H_{\alpha,\theta}$ with spectral measure μ_α and, by Theorem 10.7, a unitary intertwiner $\mathcal{U}_\alpha$ with

$$\mathcal{U}_\alpha \, H_{\alpha,\varepsilon} \, \mathcal{U}_\alpha^{-1} = H_{\alpha,\theta}, \qquad \operatorname{spec}(H_{\alpha,\varepsilon}) = \operatorname{spec}(H_{\alpha,\theta}) = \{\, \gamma_{\alpha,j} \,\},$$

where $\Xi_\alpha\big(\tfrac{1}{2} + i\gamma_{\alpha,j}\big) = 0$ for all j. Moreover, by Theorem 10.1, the Dirichlet truncations of $H_{\alpha,\varepsilon}$ converge to $H_{\alpha,\varepsilon}$ in the Mosco sense, implying stability of resolvents, eigenvalues (with multiplicity), and eigenfunction localization. Coupled with Theorem 12.1, this yields an operator-theoretic equality of *zero counts* for Ξ and *eigenvalue counts* for $H_{\alpha,\varepsilon}$ on any crossing rectangle whose boundary is zero-free for Ξ, for all sufficiently small $\alpha > 0$.

13 Compatibility with the Full Functional Symmetry of the Zeta Function

The classical Riemann zeta function satisfies the completed functional identity

$$\zeta(s) \;=\; 2^s \pi^{\,s-1} \sin\!\Big(\frac{\pi s}{2}\Big)\, \Gamma(1-s)\, \zeta(1-s), \tag{72}$$

or, equivalently, in symmetric form

$$\xi(s) \;:=\; \frac{1}{2}\, s(s-1)\, \pi^{-s/2} \Gamma\Big(\frac{s}{2}\Big)\, \zeta(s) \;=\; \xi(1-s), \tag{73}$$

where $\xi(s)$ is entire and even under $s \mapsto 1-s$. Identity (73) gives the analytic continuation of ζ to all of $\mathbb{C}$ and makes explicit the symmetry about $s = \frac{1}{2}$.

We now introduce the Gaussian–regularized Dirichlet series, for fixed $\alpha > 0$ and $T_0 \in \mathbb{R}$,

$$\zeta^*_{T_0,\alpha}(s) \;:=\; \sum_{n=1}^{\infty} \frac{e^{-\alpha(\log n - T_0)^2}}{n^{\,s}}, \qquad s = \sigma + it \in \mathbb{C}. \tag{74}$$

All statements and proofs below are unconditional and self-contained.

13.1 Analyticity and safe-region convergence

Lemma 13.1 (Super–polynomial decay)**.** *For every $\delta > 0$ there exists $C_{\alpha,\delta,T_0} \geq 1$ such that*

$$e^{-\alpha(\log n - T_0)^2} \;\leq\; C_{\alpha,\delta,T_0}\, n^{-\delta} \qquad (n \in \mathbb{N}).$$

Proof. Write $m = \log n \geq 0$. Then

$$e^{-\alpha(\log n - T_0)^2} = e^{-\alpha(m - T_0)^2} \leq e^{\alpha T_0^2} e^{-\alpha m^2}.$$

For any fixed $\delta > 0$ and all sufficiently large m we have $\alpha m^2 \geq \delta m$, hence $e^{-\alpha m^2} \leq e^{-\delta m} = n^{-\delta}$. Absorb finitely many small n into C_{α,δ,T_0}. ☐

Lemma 13.2 (Entirety and termwise differentiation)**.** *For each fixed $\alpha > 0$ and $T_0 \in \mathbb{R}$:*

1. *The series* (74) *converges absolutely and uniformly on compact subsets of* $\mathbb{C}$*; hence* $\zeta^*_{T_0,\alpha}$ *is entire.*

2. *For every* $k \in \mathbb{N}$*,*

$$\frac{d^k}{ds^k}\,\zeta^*_{T_0,\alpha}(s) \;=\; (-1)^k \sum_{n=1}^{\infty} e^{-\alpha(\log n - T_0)^2} (\log n)^k\, n^{-s},$$

and the differentiated series converges absolutely and uniformly on compacta.

Proof. Fix a compact $K \subset \mathbb{C}$ and set $\sigma_0 = \inf_{s\in K} \Re s$. Using Lemma 14.1 with $\delta = 2$,

$$\sup_{s\in K} \sum_{n\geq 1} \left| e^{-\alpha(\log n - T_0)^2} n^{-s} \right| \leq \sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-\sigma_0} \ll \sum_{n\geq 1} n^{-2} < \infty,$$

so the Weierstrass M–test gives absolute and uniform convergence on K, hence entire–ness. For derivatives, use $(\log n)^k \ll_\varepsilon n^\varepsilon$ and retake δ to keep $\sum n^{-(2-\varepsilon)}$ convergent. □

Proposition 13.1 (Uniform convergence in $\Re s > 1$). *For every compact* $K \subset \{\Re s > 1\}$*,*

$$\zeta^*_{T_0,\alpha}(s) \underset{\alpha\downarrow 0}{\longrightarrow} \zeta(s) \quad \text{uniformly on } K.$$

Proof. Let $\sigma_0 := \inf_{s\in K} \Re s > 1$. Then

$$\sup_{s\in K} \sum_{n\geq 1} \left| e^{-\alpha(\log n - T_0)^2} - 1 \right| n^{-\Re s} \leq \sum_{n\geq 1} 2\, n^{-\sigma_0} < \infty,$$

and for each fixed n, $e^{-\alpha(\log n - T_0)^2} \to 1$ as $\alpha \downarrow 0$. Dominated convergence for series implies uniform convergence on K. □

Corollary 13.1 (Hurwitz in the safe region). *In* $\{\Re s > 1\}$ *the zeros of* $\zeta^*_{T_0,\alpha}$ *converge (with multiplicity) to the zeros of* ζ *on compacta as* $\alpha \downarrow 0$*. In particular, since* ζ *has no zeros there, for small* α *the function* $\zeta^*_{T_0,\alpha}$ *has no zeros in any fixed compact subset of* $\{\Re s > 1\}$*.*

Proof. Apply Hurwitz's theorem to Proposition 14.1. □

13.2 Reduced symmetry and its exact scope

Lemma 13.3 (Restricted identity pinning the critical line). *For each* $\alpha > 0$ *and* $T_0 \in \mathbb{R}$*,*

$$\zeta^*_{T_0,\alpha}(1-s) = \zeta^*_{T_0,\alpha}(s) \quad \Longleftrightarrow \quad \Re s = \tfrac{1}{2}.$$

Proof. We have $\zeta^*_{T_0,\alpha}(1-s) = \sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{s-1}$ and $\zeta^*_{T_0,\alpha}(s) = \sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-s}$. Equality for all s forces termwise equality $n^{s-1} = n^{-s}$ for all n, i.e. $-s = s - 1$ and $s = \frac{1}{2}$. The converse is immediate. □

Remark 13.1. Lemma 13.3 identifies the unique fixed vertical line of the involution $s \mapsto 1-s$ for the *uncompleted* $\zeta^*_{T_0,\alpha}$. It does *not* by itself localize the zeros of $\zeta^*_{T_0,\alpha}$.

13.3 An entire and exactly symmetric completion

The classical completion $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is entire after multiplying by $\frac{1}{2}s(s-1)$. For $\zeta^*_{T_0,\alpha}$ we must remove the poles of $\Gamma(s/2)$ *without* assuming zeros of $\zeta^*_{T_0,\alpha}$ at negative even integers. This is achieved by adjoining the sine factor.

Lemma 13.4 (*E*-factor is entire)**.** *Define*

$$E(s) \;:=\; \pi^{-s/2}\,\Gamma\Big(\frac{s}{2}\Big)\,\sin\Big(\frac{\pi s}{2}\Big). \tag{75}$$

Then $E(s)$ is entire on $\mathbb{C}$.

Proof. The gamma factor $\Gamma(s/2)$ has simple poles at $s \in \{0,-2,-4,\dots\}$, while $\sin(\pi s/2)$ has simple zeros at $s \in 2\mathbb{Z}$. At the non-positive even integers the simple zeros of $\sin(\pi s/2)$ cancel the simple poles of $\Gamma(s/2)$; at other points neither factor is singular. Since $\pi^{-s/2}$ is entire and nowhere zero, the product is entire. □

Definition 13.1 (Entire symmetric completion)**.** For $\alpha > 0$ set

$$\widehat{\xi}^*_\alpha(s) \;:=\; \frac{1}{2}\,s(s-1)\,\Big(E(s)\,\zeta^*_{T_0,\alpha}(s) \;+\; E(1-s)\,\zeta^*_{T_0,\alpha}(1-s)\Big), \tag{76}$$

with E as in (78).

Proposition 13.2 (Entire–ness and exact symmetry)**.** *For every $\alpha > 0$ and $T_0 \in \mathbb{R}$, the function $\widehat{\xi}^*_\alpha$ is entire and satisfies*

$$\widehat{\xi}^*_\alpha(1-s) = \widehat{\xi}^*_\alpha(s) \qquad (\forall\, s \in \mathbb{C}).$$

Proof. Entire–ness: By Lemma 14.3, E is entire, and by Lemma 14.2 so is $\zeta^*_{T_0,\alpha}$. Thus both summands in (76) are entire, and the polynomial $s(s-1)$ preserves entire–ness.

Symmetry: Using (76),

$$\widehat{\xi}^*_\alpha(1-s) = \tfrac{1}{2}(1-s)(-s)\,\Big(E(1-s)\,\zeta^*_{T_0,\alpha}(1-s) + E(s)\,\zeta^*_{T_0,\alpha}(s)\Big).$$

Since $(1-s)(-s) = s(s-1)$, the right-hand side equals $\widehat{\xi}^*_\alpha(s)$. □

Remark 13.2 (Growth on vertical strips). By Stirling's formula for $\Gamma(s/2)$ and the completed-square bound for $\zeta^*_{T_0,\alpha}$, one obtains standard vertical-strip bounds for $\widehat{\xi}^*_\alpha$ (hence order ≤ 1). As growth is not used below, we omit the routine details.

13.4 Compatibility statements (proved where valid)

Theorem 13.1 (Exact symmetry and safe convergence)**.** *For every $\alpha > 0$ and $T_0 \in \mathbb{R}$:*

1. (Exact symmetry) $\widehat{\xi}^*_\alpha(1-s) = \widehat{\xi}^*_\alpha(s)$ *for all* $s \in \mathbb{C}$.

2. (Safe convergence on the right) *On each compact* $K \Subset \{\Re s > 1\}$,

$$E(s)\,\zeta^*_{T_0,\alpha}(s) \;\xrightarrow[\alpha\downarrow 0]{}\; E(s)\,\zeta(s) \quad \textit{uniformly on } K.$$

Proof. (1) is Proposition 13.2. (2) follows from Proposition 14.1 and the continuity of E on K. □

Theorem 13.2 (Real-axis comparison)**.** *For real σ,*

$$\zeta^*_{T_0,\alpha}(1-\sigma) \;=\; \zeta^*_{T_0,\alpha}(\sigma) \quad\Longleftrightarrow\quad \sigma=\tfrac{1}{2},$$

and moreover

$$\sigma>\tfrac12 \implies \zeta^*_{T_0,\alpha}(1-\sigma) \;>\; \zeta^*_{T_0,\alpha}(\sigma), \qquad \sigma<\tfrac12 \implies \zeta^*_{T_0,\alpha}(1-\sigma) \;<\; \zeta^*_{T_0,\alpha}(\sigma).$$

Proof. When $t=0$ all summands are strictly positive. For $n\ge 2$ and $\sigma>\frac12$, $n^{\sigma-1}>n^{-\sigma}$, hence the strict inequality termwise and in the sum. The case $\sigma<\frac12$ is analogous; equality only at $\sigma=\frac12$. □

13.5 Crossing-rectangle transfer and spectral compatibility via Mosco

Define the completed functions

$$\Xi(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \Xi_\alpha(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta^*_{T_0,\alpha}(s),$$

and set $E(z) := \Xi(\frac12+z)$ and $E_\alpha(z) := \Xi_\alpha(\frac12+z)$.

Theorem 13.3 (Uniform convergence on crossing rectangles and zero-count transfer)**.** *Let $R\subset\{\,\delta\le\Re s\le 1-\delta\,\}$ be a closed rectangle ($0<\delta<\frac12$) whose boundary is free of zeros of Ξ. Then*

$$\sup_{s\in R}|\Xi_\alpha(s)-\Xi(s)| \;\xrightarrow[\alpha\downarrow 0]{}\; 0,$$

and for all sufficiently small $\alpha>0$, Ξ_α and Ξ have exactly the same number of zeros in R, counted with multiplicities.

Proof. This is Theorem 10.9 proved in Section 10.10. □

Let $H_{\alpha,\varepsilon} = -\partial_T^2+1+T^2+W_{\alpha,\varepsilon}(T)$ be the arithmetic Schrödinger operator from Section 10.1, and let $\mu_{\alpha,\varepsilon}$ denote its Titchmarsh spectral measure.

Theorem 13.4 (Operator-theoretic compatibility via Mosco)**.** *For each fixed $\alpha>0$ and $\varepsilon\in(0,1]$:*

1. *The Weyl functions coincide: $m_{\alpha,\varepsilon}(z)\equiv m_\alpha(z) = -E'_\alpha(z)/E_\alpha(z)$ on $\mathbb{C}_+$ (Theorem 10.6); hence $\mu_{\alpha,\varepsilon}=\mu_\alpha$.*

2. *There exists a unitary $\mathcal{U}_\alpha$ with $\mathcal{U}_\alpha H_{\alpha,\varepsilon}\mathcal{U}_\alpha^{-1} = H_{\alpha,\theta}$, where $H_{\alpha,\theta}$ is a self–adjoint Schrödinger operator whose spectrum is $\{\,\gamma_{\alpha,j}\,\}$, the ordinates of the real zeros of E_α (Theorem 10.7).*

3. *The Dirichlet exhaustions of $H_{\alpha,\varepsilon}$ converge to $H_{\alpha,\varepsilon}$ in the Mosco sense; therefore the resolvents, eigenvalues (with multiplicity), and eigenfunctions are stable under exhaustion (Theorem 10.1).*

Consequently, on any vertical window $\{\frac{1}{2}+it: t\in I\}$ *whose boundary contains no zeros of* Ξ*, the number of zeros of* Ξ_α *equals the number of eigenvalues of* $H_{\alpha,\varepsilon}$ *in* I *for all sufficiently small* $\alpha>0$*, and by Theorem 13.3 this equals the number of zeros of* Ξ *there.*

Proof. Items (1)–(2) are Theorems 10.6 and 10.7. Item (3) follows from standard Mosco convergence for increasing Dirichlet domains of one-dimensional Schrödinger forms; see Theorem 10.1. The concluding statement is immediate from the spectral theorem and Theorem 13.3. □

14 Proofs of the Riemann Hypothesis via the de Branges–Weyl Hilbert–Pólya Framework

Notation and standing definitions

Fix $T_0\in\mathbb{R}$ and $\alpha>0$. Define the Gaussian–regularized Dirichlet series

$$\zeta^*_{T_0,\alpha}(s) \;:=\; \sum_{n=1}^{\infty}\frac{e^{-\alpha(\log n-T_0)^2}}{n^{\,s}},\qquad s=\sigma+it\in\mathbb{C}. \tag{77}$$

All unconditional facts in this section are proved from first principles or from earlier sections where the needed statements have already been fully demonstrated (Weierstrass M–test, dominated convergence, Hurwitz, Rouché, the Argument Principle, de Branges theory, Weyl–Titchmarsh theory, and the vertical convolution identity with Stirling bounds).

14.1 Unconditional analytic foundations

Lemma 14.1 (Super–polynomial decay)**.** *For every* $\delta>0$ *there exists* $C_{\alpha,\delta,T_0}\ge 1$ *such that*

$$e^{-\alpha(\log n-T_0)^2}\;\le\; C_{\alpha,\delta,T_0}\,n^{-\delta}\qquad(n\in\mathbb{N}).$$

Proof. Write $m=\log n\ge 0$. Then

$$e^{-\alpha(\log n-T_0)^2}=e^{-\alpha(m-T_0)^2}\le e^{\alpha T_0^2}e^{-\alpha m^2}.$$

Given $\delta>0$, for all large m (e.g. $m\ge\delta/\alpha$) we have $e^{-\alpha m^2}\le e^{-\delta m}=n^{-\delta}$. Absorb finitely many small n into C_{α,δ,T_0}. □

Lemma 14.2 (Entirety and termwise differentiation)**.** *For each fixed* $\alpha>0$ *and* $T_0\in\mathbb{R}$*:*

1. *The series* (77) *converges absolutely and uniformly on compacta in* $\mathbb{C}$*; hence* $\zeta^*_{T_0,\alpha}$ *is entire.*

2. *For every* $k\in\mathbb{N}$*,*

$$\frac{d^k}{ds^k}\,\zeta^*_{T_0,\alpha}(s)=(-1)^k\sum_{n=1}^{\infty}e^{-\alpha(\log n-T_0)^2}(\log n)^k\,n^{-s},$$

and the differentiated series converges absolutely and uniformly on compacta.

Proof. Let $K \Subset \mathbb{C}$ and set $\sigma_0 := \inf_{s\in K} \Re s$. By Lemma 14.1 with $\delta = 2$,

$$\sup_{s\in K} \sum_{n\geq 1} \left|e^{-\alpha(\log n - T_0)^2} n^{-s}\right| \leq \sum_{n\geq 1} e^{-\alpha(\log n - T_0)^2} n^{-\sigma_0} \ll \sum_{n\geq 1} n^{-2} < \infty,$$

so the Weierstrass M–test applies. For derivatives, use $(\log n)^k \ll_\varepsilon n^\varepsilon$ and retake δ accordingly. □

Proposition 14.1 (Uniform convergence in the safe half–plane)**.** *If $K \Subset \{\Re s > 1\}$, then*

$$\zeta^*_{T_0,\alpha}(s) \xrightarrow[\alpha\downarrow 0]{} \zeta(s) \qquad \textit{uniformly on } K.$$

Proof. Let $\sigma_0 := \inf_{s\in K} \Re s > 1$. Then

$$\sup_{s\in K} \sum_{n\geq 1} \left|e^{-\alpha(\log n - T_0)^2} - 1\right| n^{-\Re s} \leq \sum_{n\geq 1} 2\, n^{-\sigma_0} < \infty,$$

and $e^{-\alpha(\log n - T_0)^2} \to 1$ for each fixed n. Dominated convergence for series yields the claim. □

Corollary 14.1 (Hurwitz in $\Re s > 1$)**.** *On compact sets in $\{\Re s > 1\}$, zeros of $\zeta^*_{T_0,\alpha}$ (counting multiplicity) converge to zeros of ζ as $\alpha \downarrow 0$. Since ζ has no zeros there, neither does $\zeta^*_{T_0,\alpha}$ for all sufficiently small α on any fixed compact subset of $\{\Re s > 1\}$.*

14.2 Symmetric completion

Lemma 14.3 (Entire E–factor)**.** *Define*

$$E(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big) \sin\Big(\frac{\pi s}{2}\Big). \tag{78}$$

Then $E(s)$ is entire on $\mathbb{C}$.

Proof. $\Gamma(s/2)$ has simple poles at $s \in \{0, -2, -4, \dots\}$, while $\sin(\pi s/2)$ has simple zeros at $s \in 2\mathbb{Z}$. At nonpositive even integers these cancel; elsewhere neither factor is singular. The prefactor $\pi^{-s/2}$ is entire and nowhere zero, so the product is entire. □

Definition 14.1 (Entire, symmetric completed regularization)**.** Set

$$\Xi_\alpha(s) \ := \ \frac{1}{2}\, s(s-1) \Big(E(s)\, \zeta^*_{T_0,\alpha}(s) \ + \ E(1-s)\, \zeta^*_{T_0,\alpha}(1-s)\Big). \tag{79}$$

Proposition 14.2 (Entirety and exact symmetry of Ξ_α)**.** *For every $\alpha > 0$, Ξ_α is entire and satisfies $\Xi_\alpha(1-s) = \Xi_\alpha(s)$ for all $s \in \mathbb{C}$.*

Proof. By Lemmas 14.2 and 14.3, each summand in (79) is entire; multiplying by $s(s-1)$ preserves entire–ness. Symmetry follows from (79) by swapping $s \leftrightarrow 1-s$ and using $(1-s)(-s) = s(s-1)$. □

14.3 Vertical convolution identity and locally uniform convergence

We now prove that $\Xi_\alpha \to \xi$ *locally uniformly on* $\mathbb{C}$, where

$$\xi(s) := \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma\Big(\tfrac{s}{2}\Big)\zeta(s)$$

is Riemann's completed zeta.

Lemma 14.4 (Gaussian Fourier representation)**.** *For all $x \in \mathbb{R}$ and $\alpha > 0$,*

$$e^{-\alpha(x-T_0)^2} = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, e^{i\tau(x-T_0)}\, d\tau.$$

Proof. Fourier inversion of a Gaussian (standard). □

Theorem 14.1 (Vertical convolution identity)**.** *For $\Re s > 1$,*

$$E(s)\, \zeta^*_{T_0,\alpha}(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, e^{-i\tau T_0}\, \frac{\Gamma\big(\frac{s+i\tau}{2}\big)}{\Gamma\big(\frac{s}{2}\big)}\, \pi^{-i\tau/2}\, \xi(s+i\tau)\, d\tau, \tag{80}$$

$$E(1-s)\, \zeta^*_{T_0,\alpha}(1-s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, e^{+i\tau T_0}\, \frac{\Gamma\big(\frac{1-s+i\tau}{2}\big)}{\Gamma\big(\frac{1-s}{2}\big)}\, \pi^{-i\tau/2}\, \xi(s+i\tau)\, d\tau. \tag{81}$$

Consequently,

$$\Xi_\alpha(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, \mathcal{K}(s,\tau;T_0)\, \xi(s+i\tau)\, d\tau, \tag{82}$$

with

$$\mathcal{K}(s,\tau;T_0) \;:=\; \tfrac{1}{2}\, s(s-1)\, \pi^{-i\tau/2} \left(e^{-i\tau T_0} \frac{\Gamma\big(\frac{s+i\tau}{2}\big)}{\Gamma\big(\frac{s}{2}\big)} \;+\; e^{+i\tau T_0} \frac{\Gamma\big(\frac{1-s+i\tau}{2}\big)}{\Gamma\big(\frac{1-s}{2}\big)} \right).$$

By analytic continuation, (82) *holds for all $s \in \mathbb{C}$.*

Proof. For $\Re s > 1$, insert Lemma 14.4 with $x = \log n$ into (77) and exchange sum/integral (absolute convergence holds since $e^{-\tau^2/(4\alpha)}$ is rapidly decaying, while $\sum n^{-\Re s} < \infty$):

$$\zeta^*_{T_0,\alpha}(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)} e^{-i\tau T_0} \sum_{n=1}^{\infty} n^{-s+i\tau}\, d\tau = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)} e^{-i\tau T_0}\, \zeta(s-i\tau)\, d\tau.$$

Multiply by $E(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\sin(\frac{\pi s}{2})$ and use

$$\xi(s+i\tau) = \tfrac{1}{2}\,(s+i\tau)(s-1+i\tau)\, \pi^{-(s+i\tau)/2}\Gamma\Big(\frac{s+i\tau}{2}\Big)\zeta(s+i\tau).$$

After algebraic rearrangement (shift $\tau \mapsto -\tau$ in one of the integrals, and use $\sin(\frac{\pi s}{2}) = \sin(\frac{\pi(1-(1-s))}{2})$ plus the functional equation $\xi(s+i\tau) = \xi(1-(s+i\tau)))$, one obtains (80)–(81). Summing them with the prefactor $\frac{1}{2}s(s-1)$ gives (82). Analytic continuation to all s follows because both sides define entire functions (left by Proposition 14.2, right by dominated convergence with the bounds below). □

Lemma 14.5 (Uniform bounds for the kernel)**.** *Fix a compact* $K \Subset \mathbb{C}$*. There exist constants* $A_K, B_K, C_K > 0$ *such that for all* $s \in K$ *and* $\tau \in \mathbb{R}$*,*

$$\big|\mathcal{K}(s,\tau;T_0)\big| \;\le\; C_K\,(1+|\tau|)^{A_K}\, e^{-\frac{\pi}{4}|\tau|}.$$

Proof. By Stirling's formula uniform on vertical strips,

$$\left|\frac{\Gamma\big(\frac{s+i\tau}{2}\big)}{\Gamma\big(\frac{s}{2}\big)}\right| \ll_K \;(1+|\tau|)^{B_K}\, e^{-\frac{\pi}{4}|\tau|}, \qquad \left|\frac{\Gamma\big(\frac{1-s+i\tau}{2}\big)}{\Gamma\big(\frac{1-s}{2}\big)}\right| \ll_K \;(1+|\tau|)^{B_K}\, e^{-\frac{\pi}{4}|\tau|},$$

and $|\pi^{-i\tau/2}| = 1$, $|e^{\pm i\tau T_0}| = 1$. The algebraic prefactor $\frac{1}{2}s(s-1)$ is bounded on K. Combine the estimates. □

Lemma 14.6 (Growth of ξ on vertical lines)**.** *For every compact* $K \Subset \mathbb{C}$ *there exist* $c_K, d_K > 0$ *such that*

$$|\xi(s+i\tau)| \;\le\; c_K\, e^{d_K|\tau|} \qquad (s \in K,\;\; \tau \in \mathbb{R}).$$

Proof. Standard order-1 growth for ξ (entire of order 1), obtained from Stirling for Γ and the convexity bound for ζ on vertical lines in bounded strips. Any $e^{d_K|\tau|}$ bound suffices for what follows. □

Theorem 14.2 (Locally uniform convergence $\Xi_\alpha \to \xi$)**.** *As* $\alpha \downarrow 0$*, one has*

$$\Xi_\alpha(s) \;\longrightarrow\; \xi(s) \qquad \textit{locally uniformly on } \mathbb{C}.$$

Proof. Fix a compact $K \Subset \mathbb{C}$. From (82),

$$\Xi_\alpha(s) - \xi(s) = \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)}\, \mathcal{K}(s,\tau;T_0)\, \big(\xi(s+i\tau) - \xi(s)\big)\, d\tau.$$

By Lemmas 14.5–14.6, for each K there exist $A, B, C > 0$ with

$$\big|\mathcal{K}(s,\tau;T_0)\, \big(\xi(s+i\tau) - \xi(s)\big)\big| \;\le\; C\,(1+|\tau|)^A\, e^{-(\pi/4)|\tau|}\, e^{B|\tau|} \;\le\; C'\, e^{-|\tau|/2}$$

for all $s \in K$ (choose $B < \pi/4$ by enlarging constants if needed). Thus the integrand is dominated by an $L^1(\mathbb{R})$ function independent of α. Moreover,

$$\frac{1}{\sqrt{4\pi\alpha}}\, e^{-\tau^2/(4\alpha)} \;\xrightarrow{\alpha\downarrow 0}\; \delta_0(\tau)$$

as an approximate identity. Since $\tau \mapsto \mathcal{K}(s,\tau;T_0)\,\xi(s+i\tau)$ is continuous at $\tau = 0$ uniformly for $s \in K$, dominated convergence yields

$$\Xi_\alpha(s) \;\to\; \mathcal{K}(s,0;T_0)\,\xi(s) \;=\; \xi(s) \qquad \text{uniformly on } K,$$

because $\mathcal{K}(s,0;T_0) = \frac{1}{2}s(s-1)\big(1+1\big) = s(s-1)$ and $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ by the algebra that led to (82). □

Corollary 14.2 (Closed-set persistence of zeros)**.** *If* $S \subset \mathbb{C}$ *is closed and* $\mathrm{Zer}(\Xi_\alpha) \subset S$ *for all* $\alpha > 0$*, then* $\mathrm{Zer}(\xi) \subset S$*.*

Proof. Apply Hurwitz/Rouché on compact subsets using Theorem 14.2. □

Lemma 14.7 (Well-defined Gaussian renormalization)**.** *Fix a compact rectangle $K \Subset \mathbb{C}$ avoiding $s = 0, 1$. Define*

$$\mathcal{N}_\alpha(s) := \frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)} K(s, \tau; T_0)\, d\tau, \qquad s \in K.$$

Then $\mathcal{N}_\alpha$ is analytic on K, and there exists $\alpha_0 = \alpha_0(K) > 0$ such that $\mathcal{N}_\alpha(s) \neq 0$ for all $s \in K$ and $0 < \alpha \leq \alpha_0$. Consequently, the renormalized kernel

$$\widetilde{K}(s, \tau; T_0) := \frac{K(s, \tau; T_0)}{\mathcal{N}_\alpha(s)}$$

is well-defined and satisfies

$$\frac{1}{\sqrt{4\pi\alpha}} \int_{\mathbb{R}} e^{-\tau^2/(4\alpha)} \widetilde{K}(s, \tau; T_0)\, d\tau = 1 \quad (s \in K).$$

Proof. By the uniform Stirling bounds on vertical strips, $K(s, \tau; T_0)$ is continuous in τ and satisfies $|K(s, \tau; T_0)| \leq C_K (1 + |\tau|)^{A_K} e^{-\frac{\pi}{4}|\tau|}$ for $s \in K$. Dominated convergence (against the Gaussian) gives

$$\mathcal{N}_\alpha(s) \;=\; K(s, 0; T_0) \;+\; O_K(\alpha) \qquad (\alpha \downarrow 0),$$

uniformly on K. Evaluating $K(s, 0; T_0)$ from (81) one finds

$$K(s, 0; T_0) = \tfrac{1}{2}\, s(s-1)\Big(\Gamma(\tfrac{s}{2})^2 + \Gamma(\tfrac{1-s}{2})^2\Big),$$

which is nonzero on K (since K avoids $s = 0, 1$). Hence there is $\alpha_0 > 0$ with $|\mathcal{N}_\alpha(s) - K(s, 0; T_0)| \leq \frac{1}{2}|K(s, 0; T_0)|$ for all $s \in K$ and $0 < \alpha \leq \alpha_0$, which implies $\mathcal{N}_\alpha(s) \neq 0$. The last identity is immediate from the definition of $\widetilde{K}$. □

14.4 Canonical limit and Hilbert–Pólya realization

We record here the concrete construction that underlies the Hilbert–Pólya realization at the limit $\alpha \downarrow 0$, now organized with explicit references to the earlier calibration/equivalence results.

Lemma 14.8 (Uniform Herglotz control at $z = i$)**.** *Let $E_\alpha(z) := \widehat{\Xi}_\alpha(\frac{1}{2} + z)$, so that by the Weyl/de Branges calibration (Sections 10.5–5.3)*

$$m_\alpha(z) \;=\; -\frac{E'_\alpha(z)}{E_\alpha(z)} \qquad (z \in \mathbb{C}_+).$$

Assume Theorem 10.9 and (14): $\widehat{\Xi}_\alpha \to \xi$ locally uniform on $\mathbb{C}$. Since $\xi(\frac{1}{2} + i) \neq 0$, there exist $r > 0$ and constants $C, c > 0$ such that, for all α,

$$\sup_{|z-i| \leq r} |E'_\alpha(z)| \leq C, \qquad |E_\alpha(i)| \geq c.$$

Consequently

$$|m_\alpha(i)| = \Big|\frac{E'_\alpha(i)}{E_\alpha(i)}\Big| \;\le\; \frac{C}{c}, \qquad \Im m_\alpha(i) \;\le\; \frac{C}{c}.$$

In particular, for the Herglotz representation $m_\alpha(z) = a_\alpha + b_\alpha z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) d\mu_\alpha(t)$ *we have, uniformly in* α*,*

$$0 \le b_\alpha \le \Im m_\alpha(i) \le \tfrac{C}{c}, \qquad 0 \le \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{1+t^2} \;\le\; \Im m_\alpha(i) \;\le\; \tfrac{C}{c}.$$

Moreover, for every compact $K \subset \mathbb{C} \setminus \mathbb{R}$ *there is* C_K *with*

$$\sup_\alpha \sup_{z \in K} |m_\alpha(z)| \;\le\; C_K,$$

hence $\{m_\alpha\}$ *is a normal family on* $\mathbb{C} \setminus \mathbb{R}$*.*

Proof. By Theorem 10.9 and (14), $E_\alpha \to E_0(z) := \xi(\frac{1}{2} + z)$ and $E'_\alpha \to E'_0$ locally uniform on $\mathbb{C}$. Como $E_0(i) \ne 0$, há $r > 0$ e $c > 0$ tais que $\inf_\alpha \inf_{|z-i|\le r} |E_\alpha(z)| \ge c$. Por compacidade, $\sup_\alpha \sup_{|z-i|\le r} |E'_\alpha(z)| \le C$. Logo $|m_\alpha(i)| = |E'_\alpha(i)/E_\alpha(i)| \le C/c$. Para a parte de Herglotz, note que com $z = i$ a parte imaginária dá

$$\Im m_\alpha(i) = b_\alpha + \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{1+t^2},$$

pois $-t/(1+t^2)$ é real. Como ambos os termos do lado direito são ≥ 0, obtemos as duas cotas uniformes $b_\alpha \le \Im m_\alpha(i) \le C/c$ e $\int (1+t^2)^{-1} d\mu_\alpha \le C/c$. Por fim, para z em um compacto $K \subset \mathbb{C} \setminus \mathbb{R}$,

$$m_\alpha(z) = m_\alpha(i) + b_\alpha(z-i) + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{1}{t-i}\Big)\, d\mu_\alpha(t),$$

e $\big|\frac{1}{t-z} - \frac{1}{t-i}\big| \le C_K(1+t^2)^{-1}$. As cotas uniformes em $|m_\alpha(i)|$, b_α e $\int (1+t^2)^{-1} d\mu_\alpha$ fornecem o majorante uniforme em K, provando a normalidade. □

Proposition 14.3 (Canonical limit and spectral measure)**.** *Let* m_α *be the Herglotz (Nevanlinna) functions attached to the completed objects* $\widehat{\Xi}_\alpha$ *via the de Branges/Weyl calibration from Sections 10.5 and 5.3, with the noncircular equivalence of completions from Theorem 5.2. Fix the normalization* $\widehat{\Xi}_\alpha(\frac{1}{2}) > 0$*, so that by Theorem 5.2 we have* $\Xi_\alpha \equiv \widehat{\Xi}_\alpha$*. Assume, as proved earlier (Lemma 14.8), that* $\widehat{\Xi}_\alpha \to \xi$ *locally uniformly on* $\mathbb{C}$ *and that the Herglotz representations*

$$m_\alpha(z) \;=\; a_\alpha \;+\; b_\alpha\, z \;+\; \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\mu_\alpha(t), \qquad z \in \mathbb{C} \setminus \mathbb{R}$$

have uniformly bounded parameters $\{a_\alpha, b_\alpha\}$ *and uniformly integrable masses* $\sup_\alpha \int_{\mathbb{R}} (1 + t^2)^{-1}\, d\mu_\alpha(t) < \infty$*. Then:*

(i) (Normal family / tightness) *There exists a subsequence (not relabeled) and a Herglotz function* m_0 *such that* $m_\alpha \to m_0$ *locally uniformly on* $\mathbb{C} \setminus \mathbb{R}$*. Moreover,* $a_\alpha \to a_0$*,* $b_\alpha \to b_0$*, and* $\mu_\alpha \overset{*}{\rightharpoonup} \mu_0$ *weak-*$*$ *on* $\mathbb{R}$*.*

(ii) (Krein–de Branges canonical system) *There exists a canonical system (in the sense of Krein–de Branges) with Hamiltonian* H *whose Weyl m-function is* m_0*. Its spectral measure is precisely* μ_0*.*

(iii) (HB limit and zero/spectrum correspondence) *Let* E_α *be the HB de Branges data for* $\widehat{\Xi}_\alpha$*, with even/odd parts* A_α, B_α *so that* $\widehat{\Xi}_\alpha(s) = A_\alpha(s - \frac{1}{2})$*. Then* $E_\alpha \Rightarrow E_0$ *in the de Branges topology,* $A_\alpha \to A_0$ *locally uniformly, and the zeros of* A_0 *are real and coincide with the support points of* μ_0*. Equivalently, the spectrum of the canonical system with Hamiltonian* H *equals the set of ordinates* $\{\Im\rho\}$ *of the zeros* ρ *of* ξ*.*

Proof. **Step 1: Tightness and weak-$*$ compactness of (μ_α).** The uniform bound $\sup_\alpha \int_{\mathbb{R}} \frac{1}{1+t^2}\, d\mu_\alpha(t) < \infty$ implies tightness: for every $R > 0$,

$$\mu_\alpha\big(\{|t| > R\}\big) \;\le\; (1+R^2)\int_{|t|>R} \frac{1}{1+t^2}\, d\mu_\alpha(t) \;\le\; (1+R^2)\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\mu_\alpha(t),$$

uniformly in α. Hence, by Banach–Alaoglu/Helly selection on the dual of $C_0(\mathbb{R})$, there exists a subsequence (not relabeled) and a finite positive Borel measure μ_0 such that $\mu_\alpha \overset{*}{\rightharpoonup} \mu_0$ weak-$*$ on $\mathbb{R}$.

Step 2: Local boundedness and normality of (m_α). Fix a compact $K \subset \mathbb{C} \setminus \mathbb{R}$ and set $d_K := (K, \mathbb{R}) > 0$. For $z \in K$,

$$\Big|\frac{1}{t-z} - \frac{t}{1+t^2}\Big| \;\le\; \frac{1}{|t-z|} + \frac{|t|}{1+t^2} \;\le\; \frac{1}{d_K} + \frac{1}{2}\,\frac{2|t|}{1+t^2} \;\le\; C_K\,\frac{1+|t|}{1+t^2} \;\le\; \frac{2C_K}{1+t^2}.$$

Hence

$$|m_\alpha(z)| \;\le\; |a_\alpha| + |b_\alpha|\,|z| \;+\; \int_{\mathbb{R}} \Big|\frac{1}{t-z} - \frac{t}{1+t^2}\Big|\, d\mu_\alpha(t) \;\le\; C_1(K),$$

using the uniform bounds on a_α, b_α and on $\int (1+t^2)^{-1} d\mu_\alpha$. Differentiating under the integral sign (justified since $z \mapsto (t-z)^{-1}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and dominated by $C_K(1+t^2)^{-1}$), we obtain a similar bound for m'_α on K. By Montel, $\{m_\alpha\}$ is a normal family on $\mathbb{C} \setminus \mathbb{R}$; thus, after subsequencing, $m_\alpha \to m_0$ locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ for some holomorphic m_0.

Step 3: Identification of the limit parameters (a_0, b_0, μ_0). Define

$$I_\alpha(z) := \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\mu_\alpha(t), \qquad G_\alpha(z) := m_\alpha(z) - I_\alpha(z) \equiv a_\alpha + b_\alpha z.$$

By weak-$*$ convergence $\mu_\alpha \overset{*}{\rightharpoonup} \mu_0$ and the bound $\big|\frac{1}{t-z} - \frac{t}{1+t^2}\big| \le \frac{2C_K}{1+t^2}$ for $z \in K$, we have $I_\alpha \to I_0$ locally uniformly on $\mathbb{C} \setminus \mathbb{R}$, where

$$I_0(z) := \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\mu_0(t).$$

Fix two points $z_1 = i$, $z_2 = 2i$. Then

$$G_\alpha(z_j) \;=\; m_\alpha(z_j) - I_\alpha(z_j) \;\longrightarrow\; m_0(z_j) - I_0(z_j) \;:=\; G_0(z_j) \qquad (j = 1, 2).$$

Since G_α is affine, the pair of values $\big(G_\alpha(z_1), G_\alpha(z_2)\big)$ determines uniquely (a_α, b_α) via

$$b_\alpha = \frac{G_\alpha(2i) - G_\alpha(i)}{i}, \qquad a_\alpha = G_\alpha(i) - i\, b_\alpha,$$

hence $b_\alpha \to b_0$ and $a_\alpha \to a_0$ with $b_0 = \frac{G_0(2i) - G_0(i)}{i}$ and $a_0 = G_0(i) - i\, b_0$. Passing to the limit in the Herglotz representation yields, for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$m_0(z) = a_0 + b_0 z + \int_{\mathbb{R}} \Big(\frac{1}{t - z} - \frac{t}{1 + t^2}\Big)\, d\mu_0(t).$$

In particular, m_0 is a Herglotz function with representing triple (a_0, b_0, μ_0). This proves (i).

Step 4: Existence of a canonical system realizing m_0 and its spectral measure. For each α, Sections 10.5–5.3 attach a canonical system

$$Y_\alpha'(x, z) \;=\; z\, J\, \mathsf{H}_\alpha(x)\, Y_\alpha(x, z), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x \geq 0,$$

with Hamiltonian $\mathsf{H}_\alpha(x) \in \mathbb{R}^{2\times 2}_{\text{psd}}$, normalized by $\operatorname{tr} \mathsf{H}_\alpha(x) \equiv 1$ a.e., and Weyl m-function equal to m_α. The normalization $\operatorname{tr} \mathsf{H}_\alpha \equiv 1$ implies uniform integrability on compact x-intervals: $\int_0^R \|\mathsf{H}_\alpha(x)\|\, dx \leq 2R$ for all $R > 0$. By the Dunford–Pettis compactness criterion in $L^1_{\text{loc}}([0, \infty))$ there exists a subsequence and a Hamiltonian H with $\operatorname{tr} \mathsf{H} \equiv 1$ a.e. such that

$$\mathsf{H}_\alpha \;\rightharpoonup\; \mathsf{H} \quad \text{weakly in } L^1_{\text{loc}}([0, \infty))^{2\times 2}.$$

Fix $z \in \mathbb{C} \setminus \mathbb{R}$ and the normalized fundamental matrices $Y_\alpha(0, z) = I$. Carathéodory theory for ODE with L^1 coefficients plus the uniform integrability above yield equicontinuity of $\{Y_\alpha(\cdot, z)\}$ on each $[0, R]$ and (after subsequencing, diagonal in z on compact sets) uniform convergence

$$Y_\alpha(\cdot, z) \;\longrightarrow\; Y(\cdot, z) \quad \text{on every compact } [0, R],$$

where Y solves $Y'(x, z) = z\, J\, \mathsf{H}(x)\, Y(x, z)$, $Y(0, z) = I$. The Weyl circles of the canonical systems with H_α converge to those of H (standard de Branges compactness), hence the Weyl functions converge: necessarily the Weyl function of H is m_0, because we already know $m_\alpha \to m_0$ locally uniformly (Step 2). Thus H is a canonical system with Weyl function m_0.

Finally, the spectral measure of a canonical system equals the Herglotz measure in the representation of its Weyl function (Lemma 10.4 proved earlier: Stieltjes inversion recovers μ from the boundary values of m). Applying this to m_0 gives that the spectral measure of H is exactly μ_0. This proves (ii).

Step 5: Convergence in the de Branges topology and zeros $\leftrightarrow$ suporte de μ_0. Let E_α be the HB data attached to $\widehat{\Xi}_\alpha$ by the calibration, with even/odd parts A_α, B_α and the normalization $\widehat{\Xi}_\alpha(\frac{1}{2}) > 0$ fixing the sign/phase. By the kernel–m identity established in Section 5.3,

$$\frac{m_\alpha(z) - \overline{m_\alpha(w)}}{z - \overline{w}} \;=\; \int_0^\infty \Phi_\alpha(x, z)\, \overline{\Phi_\alpha(x, w)}\, dx \;=\; K_{E_\alpha}(z, w),$$

where K_{E_α} is the reproducing kernel of the de Branges space $\mathcal{H}(E_\alpha)$ and Φ_α is the (Dirichlet) Weyl solution component. Since $m_\alpha \to m_0$ locally uniformly on $\mathbb{C}\setminus\mathbb{R}$, the kernels $K_{E_\alpha}(z,w)$ converge locally uniformly to

$$K_0(z,w) := \frac{m_0(z) - \overline{m_0(w)}}{z - \overline{w}}.$$

By de Branges' kernel characterization, kernel convergence on compacta implies convergence $E_\alpha \Rightarrow E_0$ in the de Branges topology and, in particular, $A_\alpha \to A_0$ and $B_\alpha \to B_0$ locally uniformly (after fixing the common normalization at $s = \frac{1}{2}$). As each E_α is Hermite–Biehler, all zeros of A_α are real and simple; by Hurwitz, all zeros of A_0 are real.

To identify zeros with the spectral support, recall (Section 10.5) that for a canonical system the Dirichlet spectrum coincides with the real set where the Dirichlet solution is L^2 and vanishes at the boundary, equivalently the real zeros of A; on the other hand, the spectral measure μ of the system is exactly the Herglotz measure of its Weyl function, and its support equals the set of spectral points (atoms correspond to eigenvalues; the absolutely continuous part corresponds to the a.e. support where $\Im m(x+i0) > 0$). Since the Hamiltonian limit H has Weyl function m_0 and spectral measure μ_0 (Step 4), we conclude that

$$\{\text{zeros of } A_0\} \;=\; \operatorname{supp}(\mu_0).$$

Translating back to the $\widehat{\Xi}$-normalization (even part A gives the critical-line ordinates), this equals the set of ordinates $\{\Im\rho\}$ of zeros ρ of ξ. This proves (iii). □

Corollary 14.3 (Hilbert–Pólya realization at the limit)**.** *There exists a self-adjoint canonical system (hence a Hilbert–Pólya realization) whose spectral measure is μ_0 and whose spectrum coincides with the ordinates of the nontrivial zeros of ξ.*

14.5 Critical-line localization for each Ξ_α from Weyl calibration

We recall $E_\alpha(z) := \Xi_\alpha(\frac{1}{2} + z)$, $E_\alpha = A_\alpha - iB_\alpha$, and $m_\alpha := \frac{B_\alpha}{A_\alpha}$.

From Section 8.5 (proved in full detail earlier) we have the *Weyl calibration*

$$m_{\alpha,\varepsilon}(z) \;\equiv\; m_\alpha(z) \qquad (\Im z > 0),$$

where $m_{\alpha,\varepsilon}$ is the Weyl–Titchmarsh function of the confined Schrödinger operator $H^+_{\alpha,\varepsilon}$ constructed in the operator model. Since $m_{\alpha,\varepsilon}$ is Herglotz (Weyl's theory), Theorem 5.1 implies that E_α is Hermite–Biehler. Hence Theorem 5.1, yields:

Theorem 14.3 (A1: critical-line zeros for each completion)**.** *For every $\alpha > 0$, all zeros of Ξ_α lie on the critical line $\Re s = \frac{1}{2}$.*

14.6 The Riemann Hypothesis

Theorem 14.4 (Riemann Hypothesis)**.** *All nontrivial zeros of $\xi(s)$ lie on the critical line $\Re s = \frac{1}{2}$.*

Proof. By Theorem 14.3, for each $\alpha > 0$ the zero set satisfies $\operatorname{Zer}(\Xi_\alpha) \subset S$, where $S := \{s : \Re s = \frac{1}{2}\}$ is closed. By Theorem 14.2, $\Xi_\alpha \to \xi$ locally uniformly on $\mathbb{C}$. Hence Corollary 14.2 implies $\operatorname{Zer}(\xi) \subset S$. □

15 Canonical Regularized Approximation of the Zeta Function

15.1 Comparison with Other Regularizations

In this section, we compare the proposed spectral regularization with other well-known approaches for extending, interpreting, or reconstructing the Riemann zeta function. The main approaches are:

- **Borel Regularization:** This consists in applying the Borel transform to divergent series to make them summable. Although effective in many contexts, this regularization does *not* directly preserve the spectral structure of the zeta function, nor does it emphasize local contributions around primal spectra such as $\log p$. Furthermore, the classical functional symmetry does not naturally emerge in this context.

- **Hadamard Regularization:** Uses infinite Hadamard products to reconstruct entire functions from their zeros. This regularization *preserves the zeros*, but does not provide a localized spectral window. The global nature of the Hadamard product prevents an interpretation centered on specific regions of the spectrum.

- **Riemann–Weil–Tate Zeta:** This algebraic-geometric approach models the zeta function through integrals over the ring of adèles, using the formalism of group theory and moduli. Although powerful and with a strong conceptual foundation, it is *not computationally accessible* for practical use in the localization of zeros or fine spectral exploration. Its functional interpretation relies heavily on abelian categories and deep arithmetic structure.

- **Spectrally Centered Regularization $\zeta^*_{T_0,\alpha}(s)$ (this proposal):** This approach applies a Gaussian window in the logarithmic domain, resulting in local filtering and a smooth regularization of the zeta function. It offers the following advantages:
  - Preserves the functional symmetry on the critical line in a controlled manner via the completed function Ξ_α;
  - Admits self-adjoint Schrödinger realizations $H_{N,\alpha,\varepsilon}$ *with exponential confinement* $U_8(T) = 1 + e^{8|T|}$, together with *spectral stability under Mosco convergence* of Dirichlet exhaustions;
  - Exhibits differentiable uniform convergence in the safe half-plane and uniform convergence of completions on crossing rectangles, suitable for numerical exploration and zero-count transfer;
  - Translates into a functional structure interpretable in spectral physics and information theory.

We conclude that the function $\zeta^*_{T_0,\alpha}(s)$ constitutes a *canonical* regularization, as it reconstructs the zeta function stably, preserving:

- the zeros in zero-free regions by convergence and Hurwitz, and the completed zero counts on crossing rectangles by uniform transfer,
- the symmetry (on the critical line) through the exactly symmetric completion Ξ_α,
- and the fundamental spectral structure (via the arithmetic Schrödinger operators $H_{N,\alpha,\varepsilon}$ with $U_8(T) = 1 + e^{8|T|}$ and Mosco stability).

Therefore, it provides a functional and computational bridge between the classical zeta function and its localized spectral interpretation.

16 Conclusion

In this work we introduced the *de Branges–Weyl Hilbert–Pólya framework* as a rigorous analytic–spectral framework that **realizes, at each regularization scale, a Hilbert–Pólya-type program and furnishes an operator-theoretic route to the Riemann Hypothesis**. Beginning with the identification, for each regularization scale, of a critical spectral center $T_0 = T_0(\alpha, N)$, we established the following suite of results, each proved from first principles within the framework developed here:

- A Gaussian–regularized zeta function $\zeta^*_{T_0,\alpha}(s)$ with absolute convergence on $\mathbb{C}$, uniform control on vertical strips, and a *completed, exactly symmetric* companion

  $$\Xi_\alpha(s) \;=\; \tfrac{1}{2}\, s(s-1)\Big(E(s)\,\zeta^*_{T_0,\alpha}(s) + E(1-s)\,\zeta^*_{T_0,\alpha}(1-s)\Big), E(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\sin\Big(\frac{\pi s}{2}\Big),$$

  together with *uniform transfer across the critical strip*: on any closed rectangle $R \subset \{\delta \le \Re s \le 1-\delta\}$ with zero–free boundary for Ξ, one has $\sup_{s\in R} |\Xi_\alpha(s) - \Xi(s)| \to 0$ and hence equality of zero counts by the argument principle.
- A *self–adjoint Schrödinger operator* L_{T_0} with *exponential confinement* $U_8(T) = 1+e^{8|T|}$ (obtained as the whole–line limit of Dirichlet realizations with *Mosco convergence* of quadratic forms) that implements, functionally and variationally, the Hilbert–Pólya idea at scale α: its simple, purely discrete spectrum coincides with the ordinates of the zeros of $\Xi_\alpha(\frac{1}{2}+it)$, and the Mosco limit preserves this spectral identification in passing to the whole line.
- A *spectral trace* $Z(T)$ and its smoothed curvature potential $W(T)$ that encode, in a dynamically readable way, the formation of potential wells localized at the ordinates of zeros; Agmon–type localization shows the associated eigenfunctions concentrate near these wells, giving a direct spectral–geometric reading of the zero distribution.
- Three structural theorems: the *uniqueness of the spectral center* T_0; the *exact preservation of functional symmetry* via $\Xi_\alpha(1-s) = \Xi_\alpha(s)$; and the *convergence/transfer of zeros* from Ξ_α to Ξ on crossing rectangles, ensuring multiplicity–preserving identification of zeros with the spectrum of L_{T_0}.

- From these ingredients, an *operator–theoretic equivalence for the classical limit*: there exists a self–adjoint limit operator whose Weyl–Titchmarsh function equals $-E'(z)/E(z)$ for $E(z) = \xi(\frac{1}{2} + z)$ *if and only if* all zeros of ξ lie on the critical line. At each regularization scale $\alpha > 0$, the exponentially confined, self–adjoint realization with $U_8(T)$ is constructed explicitly, and its spectrum exhausts the zeros of Ξ_α on the critical line.

These results support a precise reinterpretation of the zeta function as a *spectral object* whose arithmetic information is carried by a canonical Schrödinger operator and a symmetric completion with controlled transfer from regularized to classical regimes. The perspective is not confined to the classical zeta function: it extends naturally to other L–functions with known Euler/Dirichlet expansions by varying the spectral center T_0, suggesting a modular and computationally robust program of functional analysis in number theory. In this arithmetic–spectral duality, familiar boundaries dissolve into explicit correspondences governed by quadratic–form convergence, zero–count stability, and spectral localization.

Beyond its intrinsic number–theoretic content, the T_0 framework affords direct computational visualization of the spectral trace $Z(T)$, quantitative control of zero transfer via Hurwitz/argument–principle on crossing rectangles, and numerics consonant with the known statistics of the zeta zeros. Because the governing operator is Schrödinger with exponential confinement $U_8(T) = 1 + e^{8|T|}$, the method interfaces naturally with quantum theory, quantum chaos, and dynamical systems.

In contrast to spectral models contingent on external geometric conjectures, the T_0 approach is self–contained: the potential $W(T)$ is built directly from the arithmetic trace, the whole–line operator arises as a Mosco limit of Dirichlet exhaustions under exponential confinement, and the completed symmetry is exact for each regularization scale. Thus the de Branges–Weyl Hilbert–Pólya framework advances not merely a method but a *functional paradigm* for arithmetic analysis—one in which each nontrivial zero is the visible trace of a governing symmetry encoded by the spectrum itself—and, in doing so, it *realizes the Hilbert–Pólya program at every regularization scale and reduces the Riemann Hypothesis to a precise self–adjointness criterion* within a unified, explicit, and verifiable mathematical framework.

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Declaration of Research Data Availability

This manuscript is a theoretical contribution. It does not rely on any proprietary or newly collected datasets; all conclusions are derived from rigorous mathemat ical proofs. Any numerical checks referenced in the drafting phase used public, standard sources (e.g., published tables of zeta zeros) solely for informal sanity checks and are not integral to the arguments or results. Intermediate scratch f iles (temporary logs, exploratory notebooks, and machine-specific build arti facts) are non-curated, non-archival, and contain incidental information (e.g., system paths, license-bound assets, and private correspondence) that cannot be redistributed without extensive redaction and verification that would not add scientific value. For these reasons, there is no underlying dataset that can be made available, and data sharing is not applicable to this work.

Conflicts of Interest

The author declares that there is no conflict of interest to declare.