Is the Four-Color Theorem a Real Analysis problem?

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Abstract

The Four-Color Theorem (T4C) was conjectured in 1852 and states that any planar map can be colored with four colors such that no two adjacent regions have the same color. After 125 years of attempts, the theorem was finally proven in 1977 using computational methods. All currently accepted proofs are derivatives of the original work by Appel and Haken. Despite the fact that efforts to solve this problem led to the development of new branches of mathematics, an analytical proof remains undefined.

Searching for an analytical demonstration this paper re-frames the four-color problem as a Real Analysis problem in the R^2 plane. The information contained in MAPs are expressed as a System of Equalities and Inequalities as previously did by Jansen J. U. The use of rudimentary concepts of Real Analysis and Combinatorial Analysis reduces the complexity of the analysis and conducts to a general solution of a MAP with any number of colors.

This article is an improvement of a previous one written by this author with only this reference being intentionally cited.

Introduction - A voyage through recognized and non-recognized demonstration of the Four-Color Theorem

The problem of the four colors arose in 1852 when Francis Guthrie observed that four colors seemed to be sufficient to color any map of England. After a long process his conjecture was sent to Arthur Cayley, who published the problem in the Philosophical Magazine in 1879, and then the problem gained international notoriety. This problem was maintained open until 1977 when Kenneth Appel and Wolfgang Haken published the first accepted proof that generates controversy due to the use of a computer-assisted proof in the demonstration. From 1977 until now all accepted demonstrations are based in the work of Kennet and Wolfgang.

By other side, searching aleatorily at internet there are a lot of titles involving the words “Four-Color” not of them accepted as a demonstration.

Faced to this situation it’s a challenge to present an algebraic-based proof for the Four-Color Theorem and this is the goal of this work.

Presenting the Problem

Given a (X, Y) Map as presented in Fig 1.

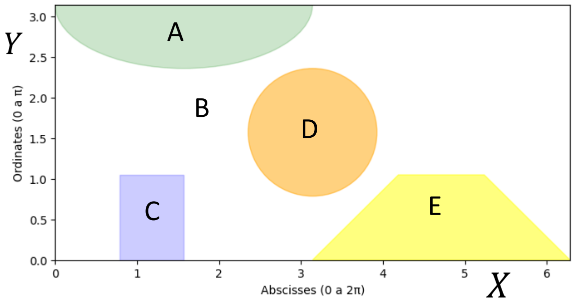

Fig 1-) Example of a (X, Y) two-dimensional plane representation of a MAP.

In Figure 1, a k enumerated coverture of X-axis parallel lines H^k can be added as presented in Figure 2.

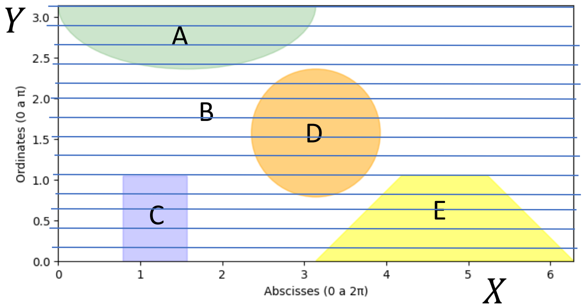

Fig 2-) Coverture of Fig 1 by H^k X-axis parallel lines.

In Figure 2, k enumerated coverture Y-axis parallel lines V^i can be added as presented in Figure 3.

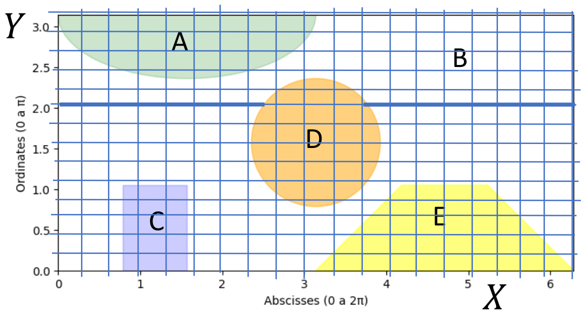

Fig 3-) Coverture of Fig 2 by V^i Y-axis parallel lines. The first and third segments, $H^k_{1(k)}$ and $H^k_{3(k)}$ are highlighted as examples of segments of H^k.

From Fig 3 it can be viewed that each horizontal line H^k contains segments delimitated by their intersections with the boundaries of each Region. Each segment is represented by $H^k_{l(k)}$ where k is the line number in ascending order and l is the numeral corresponding to the segment's position in the line k.

The $H_{l(k)}^{k}$ segments can be enumerated in increasing order, forming an ordered and enumerable set of segments.
As can be viewed in Fig 3 each V^{i} line intersects a segment $H_{l(k)}^{k}$, generating a set of points $P_{i}^{l(k)} = (V^{i} \cap H_{l(k)}^{k})$ that is also an $(n * i + k)$ ordered set and therefore constitutes an enumerable set with $n * n = n^{2}$ points.
Characteristics to be observed about the points of MAPs: there are two types of points inside a MAP: a-) inner points, the ones in the center of an open ε-ball inserted in a two-dimensional region (fractal dimension = 2) or b-) border points, the ones belonging to border lines (that are the frontiers of the regions, when they exist, with fractal dimensions <2).
Then the points $P_{i}^{l(k)}$can be inner or border points.

Rule 1) In the same way as singularities are contoured in complex calculus, an infinitesimal dislocation is applied on the frontier of every border point $P_{i}^{l(k)}$, transforming them in inner points. With this transformation the frontiers are removed of the problem in the same way as they are not considered in T4C demonstrations. With this Rule all $P_{i}^{l(k)}$ are transformed in inner points.

Analysis of the degrees of freedom (df) to color the $P_{i}^{l(k)}$ points.

Each segment $H_{l(k)}^{k}$ have two restrictions one at its left side $H_{(l-1)(k)}^{k}$ and one at its right side $H_{(l+1)(k)}^{k}$ then $P_{i}^{l(k)}$ has at maximum three df at line k written as: Color($P_{i}^{((l-1))(k)}$) ≠ Color($P_{i}^{l(k)}$), **and** Color($P_{i}^{l(k)}$) ≠ Color($P_{i}^{((l+1))(k)}$) **and** Color($P_{i}^{((l-1))(k)}$) ≠ Color($P_{i}^{((l+1))(k)}$). Then for each H^{k} three colors are enough to color the points $P_{i}^{l(k)}$ *1.

By other side border points were removed from H^{k} by Rule 1), implying that only one of de three following conditions may occur in three successive H^{k}:

a-) or Color($P_{i}^{(l^{*})(k-1)}$) ≠ Color($P_{i}^{(l)(k)}$) **and** Color($P_{i}^{(l^{**})(k+1)}$) = Color($P_{i}^{(l)(k)}$);

b-) or Color($P_{i}^{(l^{*})(k-1)}$) = Color($P_{i}^{(l)(k)}$) **and** Color($P_{i}^{(l^{**})(k+1)}$) ≠ Color($P_{i}^{(l)(k)}$) or

c-) Color($P_{i}^{(l^{*})(k-1)}$) = Color($P_{i}^{(l)(k)}$) **and** Color($P_{i}^{(l^{**})(k+1)}$) = Color($P_{i}^{(l)(k)}$).

Then the **Special Condition** Color($P_{i}^{(l^{*})(k-1)}$) ≠ Color($P_{i}^{(l)(k)}$) **and** Color($P_{i}^{(l^{**})(k+1)}$) ≠ Color($P_{i}^{(l)(k)}$) **never occur.**

Because of these conditions one more color is added resulting that four colors are enough to represent any configuration of colors of a MAP *2.

To conclude this demonstration, instead of coloring MAPs, the V_{i}^{k} points will now be colored in ascending order (order = $n * i + k$) in all possible ways using different and independent colors.

Considering that all $P_{i}^{l(k)}$ are inner points it can be stated that:

- If **l** colors (1, 2, 3, 4, 5,..**l**) are independently assigned to the V_i^k points then the set of V_i^k points has **l** ^{n*n} possible states (each state is in bijective correspondence with a number between 111...1 and **l l l** ... **l**, representing color sequences). This set of states can be divided into two subsets, one governed by the **Special Condition** and other by its complement. By construction, the states containing **Special Condition** points (isolated points) were discarded by construction applying Rule 1. The complementary set contains all viable coloring ways to color the MAP with **l** colors.
- If **five** colors are independently assigned to the V_i^k points then the set of V_i^k points has 5^{n*n} possible states, part of which represent valid MAPs and part that doesn't belong to the solution (those governed by the **Special Condition**).
- If **four** colors are independently assigned to the V_i^k points then the set of V_i^k points has 4^{n*n} possible states, part of which represent valid MAPs and part that doesn't belong to the solution (those governed by the **Special Condition**).
- If **three** colors are independently assigned to the V_i^k points then the set of V_i^k points has 3^{n*n} possible states, part of which represent valid MAPs and part that doesn't belong to the solution (those governed by the **Special Condition**). Furthermore, with 3 colors some MAPs cannot be represented. Those ones represented by four colors **cannot be represented (because there are only 3 colors disposable).**

Therefore, four colors are the minimum number of colors necessary to color any two-dimensional map.

Conclusion

Starting from standard mathematical assumptions, the Four-Color Theorem is demonstrable. The only hypothesis adopted in this work is that infinitesimal deformations can be applied to eliminate the points of the frontiers from the problem. A unified demonstration encompassing (from one to) **l** colors is derived within the same analytical framework. The enumerated coverture of MAPs by lines in an exiting idea.

This article is as simple as possible resumé of ideas. Not all possibilities are explored in the name of the concision of the work. The author hopes that the ideas presented herein signalizes the emergence of a robust analytical proof of the Four-Color Theorem. The author feels a little embarrassed that it took him wrote 4 papers before the subject gets clear. The author also acknowledges his friends and family and finalizes his efforts in this theme. In this text AI was used only in the generation of Figures.

Let's Rock and Roll!

*[1] For simplicity, the analysis including the possibility that $H_{l(k)}^{k}$ is continued at $H_{1(k+1)}^{k+1}$ and then Color($H_{l(k)}^{k}$) = Color($H_{1(k+1)}^{k+1}$) is omitted. This condition, as the ways of doing infinitesimal deformation of frontiers doesn't interfere in the demonstration at all. There may be other unconsciously adopted omissions by the author.

*[2] This conclusion is explored with other arguments in Jansen, J.U.

Conflicts of interest

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