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HyperFuzzy and SuperHyperFuzzy Group Decision-Making

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Abstract

Fuzzy sets capture vagueness by assigning each element a membership value in $[0, 1]$ [1,2]. Hyperfuzzy sets extend this idea by mapping each element to a nonempty subset of $[0, 1]$, thereby encoding both uncertainty and variability in membership degrees [3–5]. An (m, n)-superhyperfuzzy set further generalizes these notions by assigning to each nonempty member of the mth and nth iterated powersets a nonempty family of subsets of $[0, 1]$, enabling the representation of hierarchical and nested imprecision [6]. Fuzzy group decision making aggregates experts' fuzzy preference relations to produce collective rankings or to select optimal alternatives [7–9].

Despite the considerable importance of fuzzy group decision making, corresponding frameworks for hyperfuzzy and superhyperfuzzy sets remain unexplored. In this paper, we introduce Hyperfuzzy and (m, n)-SuperHyperfuzzy Group Decision Making frameworks: we define new aggregation operators and decision rules, and we illustrate their ability to accommodate richer forms of uncertainty with detailed examples.

Keywords: Fuzzy Set, Group Decision-Making, Fuzzy Group Decision-Making, HyperFuzzy Group Decision-Making, SuperHyperFuzzy Group Decision-Making

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1 Preliminaries

We now introduce the basic concepts and notation used throughout. All sets in this paper are assumed finite.

1.1 HyperFuzzy Sets and SuperHyperFuzzy Sets

Fuzzy sets model vagueness by assigning each element a membership degree in $[0, 1]$ [1,2]. Related extensions include Bipolar Fuzzy Sets [10,11], Intuitionistic Fuzzy Sets [12,13], Neutrosophic Sets [14,15], Hesitant Fuzzy Sets [16,17], and Plithogenic Sets [18–20]. In this work, we focus on two key generalizations: the *HyperFuzzy Set* [3,21,22] and its hierarchical counterpart, the *SuperHyperFuzzy Set* [4,23,24]. Both frameworks employ iterated powerset constructions to represent multiple layers of uncertainty.

Definition 1.1 (Base Set). [25] A *base set* S is the universe from which all further constructions derive:

$$S = \{ x \mid x \text{ belongs to the specified domain}\}.$$

Every element of $\mathcal{P}(S)$, $\mathcal{P}_n(S)$, and related constructs is drawn from S.

Definition 1.2 (Powerset). [26] The *powerset* of S, denoted $\mathcal{P}(S)$, is the family of all subsets of S, including $\emptyset$ and S itself:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** [27–29] Starting with $\mathcal{P}_1(H) = \mathcal{P}(H)$, the *n-th powerset* of H is defined inductively by

$$\mathcal{P}_{n+1}(H) \;=\; \mathcal{P}\big(\mathcal{P}_n(H)\big) \quad (n \geq 1).$$

Likewise, the *n-th nonempty powerset* $\mathcal{P}^*_n(H)$ omits the empty set at each stage:

$$\mathcal{P}^*_1(H) = \mathcal{P}(H) \setminus \{\emptyset\},$$

$$\mathcal{P}^*_{n+1}(H) = \mathcal{P}\big(\mathcal{P}^*_n(H)\big) \setminus \{\emptyset\}.$$

Example 1.4 (Real-world 3rd Powerset: Produce Packaging)**.** Consider the universe of fruits

$$H = \{\text{apple, banana, cherry}\}.$$

Then

$$\mathcal{P}_1(H) = \mathcal{P}(H) = \big\{\emptyset, \{\text{apple}\}, \{\text{banana}\}, \{\text{cherry}\},$$
$$\{\text{apple,banana}\}, \{\text{apple,cherry}\}, \{\text{banana,cherry}\},$$
$$\{\text{apple,banana,cherry}\}\big\}.$$

Interpret each nonempty subset as a *basket* of fruit:

$$B_1 = \{\text{apple,banana}\}, \quad B_2 = \{\text{banana,cherry}\}, \quad B_3 = \{\text{apple,cherry}\}, \;\ldots$$

Next,

$$\mathcal{P}_2(H) = \mathcal{P}\big(\mathcal{P}_1(H)\big)$$

is the family of all *pallets*, each pallet being a collection of baskets. For example:

$$P_A = \big\{B_1,\, B_2\big\} = \big\{\{\text{apple,banana}\},\ \{\text{banana,cherry}\}\big\}$$
$$\in\ \mathcal{P}_2(H),$$
$$P_B = \big\{B_3,\ \{\text{apple}\}\big\} = \big\{\{\text{apple,cherry}\},\ \{\text{apple}\}\big\}$$
$$\in\ \mathcal{P}_2(H).$$

Finally,

$$\mathcal{P}_3(H) = \mathcal{P}\big(\mathcal{P}_2(H)\big)$$

is the family of all *containers*, each container grouping several pallets. For instance:

$$C = \big\{P_A,\, P_B\big\} = \big\{\{\{\text{apple,banana}\}, \{\text{banana,cherry}\}\},\ \{\{\text{apple,cherry}\}, \{\text{apple}\}\}\big\}\ \in\ \mathcal{P}_3(H).$$

In this way, $\mathcal{P}_1(H)$ models individual baskets, $\mathcal{P}_2(H)$ models pallets of baskets, and $\mathcal{P}_3(H)$ models shipping containers of pallets—concretely illustrating the 3rd iterated powerset in a produce-packaging context.

Definition 1.5 (Fuzzy Set)**.** [1, 30] A *fuzzy set* τ on a nonempty universe Y is a function

$$\tau : Y \longrightarrow [0, 1].$$

A *fuzzy relation* δ on Y is a fuzzy subset of $Y \times Y$ satisfying

$$\delta(y, z) \;\leq\; \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 1.6 (HyperFuzzy Set)**.** [3, 31] Let X be a nonempty universe. A *HyperFuzzy Set* $\tilde{A}$ on X is given by

$$\tilde{\mu} : X \longrightarrow \tilde{\mathcal{P}}([0, 1]),$$

where $\tilde{\mathcal{P}}([0, 1])$ denotes all nonempty subsets of $[0, 1]$. For each $x \in X$, the set $\tilde{\mu}(x) \subseteq [0, 1]$ represents the range of plausible membership degrees for x, thus modeling both fuzziness and variability simultaneously.

Definition 1.7 (m, n-SuperHyperFuzzy Set)**.** [4, 32, 33] Let X be a nonempty set and let $m, n \in \mathbb{N}_0$. Define the nonempty k-th powerset of a set Y by

$$\mathcal{P}_0^*(Y) = Y, \quad \mathcal{P}_k^*(Y) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(Y)\big) \setminus \{\emptyset\}, \quad k \geq 1.$$

In particular, $\mathcal{P}_m^*(X)$ is the family of all nonempty elements of the m-th iterated powerset of X, and $\mathcal{P}_n^*([0, 1])$ is defined analogously. Then an (m, n)-SuperHyperFuzzy Set on X is a function

$$\tilde{\mu}_{m,n} : \mathcal{P}_m^*(X) \longrightarrow \tilde{\mathcal{P}}_n^*\big([0, 1]\big), \quad A \mapsto \tilde{\mu}_{m,n}(A),$$

where $\tilde{\mathcal{P}}_n^*([0, 1])$ denotes the collection of all nonempty subsets of $\mathcal{P}_n([0, 1])$. Thus each $A \in \mathcal{P}_m^*(X)$ is assigned a nonempty family of membership-degree sets $\tilde{\mu}_{m,n}(A) \subseteq \mathcal{P}_n([0, 1])$, capturing hierarchical uncertainty across both the m- and n-levels.

Example 1.8 (Real-world $(2, 2)$-SuperHyperFuzzy Set: Climate Impact Assessment)**.** Let

$$X = \{\text{Temp}, \ \text{Sea}, \ \text{Precip}\}$$

be the three key climate indicators: temperature increase (Temp), sea-level rise (Sea), and precipitation variability (Precip). We take $m = 2$, $n = 2$.

Step 1: Compute the iterated powersets.

$$\mathcal{P}_1^*(X) = \big\{\{\text{Temp}\}, \ \{\text{Sea}\}, \ \{\text{Precip}\}, \ \{\text{Temp,Sea}\}, \ \{\text{Temp,Precip}\}, \ \{\text{Sea,Precip}\}, \ \{\text{Temp,Sea,Precip}\}\big\},$$

$$\mathcal{P}_2^*(X) = \mathcal{P}\big(\mathcal{P}_1^*(X)\big) \setminus \{\emptyset\},$$

so each element of $\mathcal{P}_2^*(X)$ is a nonempty collection of subsets of X. For instance, consider

$$A = \big\{\{\text{Temp,Sea}\}, \ \{\text{Sea,Precip}\}\big\} \in \mathcal{P}_2^*(X).$$

Step 2: Define the membership mapping. We set

$$\tilde{\mu}_{2,2}(A) = \Big\{\underbrace{\{0.65, 0.75\}}_{\substack{\text{Scenario-1} \\ \text{expert-bounds}}}, \ \underbrace{\{0.80\}}_{\substack{\text{Scenario-2} \\ \text{single estimate}}}\Big\} \subseteq \tilde{\mathcal{P}}_2^*([0, 1]),$$

meaning:

- For the grouping {Temp,Sea}, experts' assessments under Scenario 1 span the interval $[0.65, 0.75]$.
- For the grouping {Sea,Precip}, Scenario 2 yields a single consensus value 0.80.

Thus

$$\tilde{\mu}_{2,2} : \mathcal{P}_2^*(X) \longrightarrow \tilde{\mathcal{P}}_2^*([0, 1])$$

assigns to each "pair-of-pairs" of indicators a nonempty family of fuzzy-degree sets, capturing both the hierarchy of indicator combinations (level 2) and nested uncertainty (level 2) arising from multiple scenarios and expert bounds.

Example 1.9 (Real-world $(1, 2)$-SuperHyperFuzzy Set: Customer Product Feedback)**.** Let

$$X = \{\text{EaseOfUse}, \ \text{Reliability}, \ \text{Design}\}, \quad m = 1, \ n = 2.$$

Then the nonempty first-level powerset is

$$\mathcal{P}_1^*(X) = \big\{\{\text{EaseOfUse}\}, \ \{\text{Reliability}\}, \ \{\text{Design}\},$$

$$\{\text{EaseOfUse, Reliability}\}, \ \{\text{EaseOfUse, Design}\},$$

$$\{\text{Reliability, Design}\}, \ \{\text{EaseOfUse, Reliability, Design}\}\big\}.$$

We define the superhyperfuzzy membership function

$$\tilde{\mu}_{1,2} : \mathcal{P}_1^*(X) \longrightarrow \tilde{\mathcal{P}}_2^*([0,1])$$

by assigning to each attribute-group a family of fuzzy-degree sets reflecting different customer segments.

For the pair

$$A = \{\text{EaseOfUse}, \text{Reliability}\},$$

set

$$\tilde{\mu}_{1,2}(A) = \Big\{ \underbrace{\{0.70,\ 0.75\}}_{\substack{\text{Casual users}\\ \text{feedback range}}},\ \underbrace{\{0.85\}}_{\substack{\text{Expert testers}\\ \text{consensus}}} \Big\} \subseteq \tilde{\mathcal{P}}_2^*([0,1]).$$

For the singleton

$$B = \{\text{Design}\},$$

set

$$\tilde{\mu}_{1,2}(B) = \Big\{ \{0.65,\ 0.70,\ 0.72\},\ \{0.80,\ 0.83\} \Big\},$$

where the first inner set is ratings from focus-group survey and the second from expert review.

Thus $\tilde{\mu}_{1,2}$ assigns to each nonempty attribute grouping in $\mathcal{P}_1^*(X)$ a nonempty collection of fuzzy-degree sets, capturing both the hierarchical grouping of product features (level 1) and the nested uncertainty across distinct customer segments (level 2).

1.2 Fuzzy Group Decision Making

Decision making involves evaluating a set of alternatives and selecting the most appropriate option based on defined criteria, stakeholder preferences, and available information [34–37]. Fuzzy Group Decision Making extends this process by aggregating multiple experts' fuzzy preference relations—mappings $R_h : X \times X \to [0,1]$—to generate collective rankings or identify robust, optimal solutions (cf. [38–41]). Related approaches include Intuitionistic Fuzzy Group Decision Making [41–44], Hesitant Fuzzy Group Decision Making [45–48], and Neutrosophic Group Decision Making [49–53].

Definition 1.10 (Fuzzy Group Decision Making)**.** (cf. [54–57]) Let $X = \{x_1, \ldots, x_n\}$ be a finite set of *alternatives* and $E = \{e_1, \ldots, e_m\}$ a finite set of *experts*. For each expert e_h, let

$$R_h : X \times X \longrightarrow [0,1]$$

be a *fuzzy preference relation* satisfying

1. reflexivity: $R_h(x_i, x_i) = \frac{1}{2}$ for all i,
2. reciprocity: $R_h(x_i, x_k) + R_h(x_k, x_i) = 1$ for all i, k.

A *fuzzy group decision making problem* is the tuple

$$\big(X,\ E,\ \{R_h\}_{h=1}^m,\ \Phi,\ D\big),$$

where

1. Φ is an *aggregation operator*

$$\Phi : \prod_{h=1}^{m} [0,1]^{X\times X} \longrightarrow [0,1]^{X\times X},$$

defined pointwise by

$$R^*(x_i, x_k) = \Phi\big(R_1(x_i, x_k), \ldots, R_m(x_i, x_k)\big), \quad \forall x_i, x_k \in X.$$

2. D is a *decision rule* assigning to the collective relation R^* either
   - a *ranking* via the fuzzy outranking $x_i \succ^* x_k \iff R^*(x_i, x_k) \geq R^*(x_k, x_i)$,
   - or a *selection* of one or more optimal alternatives, for example by computing

$$s_i = \sum_{k=1}^{n} R^*(x_i, x_k) \quad \text{and choosing all } x_i \text{ with maximal } s_i.$$

The *solution* of the problem is

$$D\big(\Phi(R_1, \ldots, R_m)\big).$$

Example 1.11 (Fuzzy Group Decision Making: Car Selection)**.** (cf. [58, 59]) Consider a committee of three experts $E = \{E_1, E_2, E_3\}$ choosing among three car models $X = \{\text{Car A}, \text{Car B}, \text{Car C}\}$. We illustrate the steps in detail.

Step 1: Expert preference relations. Each expert E_h supplies a fuzzy preference matrix

$$R_h\big(x_i, x_j\big) \in [0, 1] \quad (i, j = 1, 2, 3)$$

with $R_h(x_i, x_i) = 0.5$ and $R_h(x_i, x_j) + R_h(x_j, x_i) = 1$. Concretely:

R_1 :

	A	B	C
A	0.5	0.6	0.7
B	0.4	0.5	0.4
C	0.3	0.6	0.5

R_2 :

	A	B	C
A	0.5	0.8	0.6
B	0.2	0.5	0.55
C	0.4	0.45	0.5

R_3 :

	A	B	C
A	0.5	0.7	0.55
B	0.3	0.5	0.5
C	0.45	0.5	0.5

Step 2: Aggregation. We choose the arithmetic-mean operator

$$\Phi\big(R_1, R_2, R_3\big)(x_i, x_j) = \frac{R_1(x_i, x_j) + R_2(x_i, x_j) + R_3(x_i, x_j)}{3}.$$

Thus the collective relation $R^* = \Phi(R_1, R_2, R_3)$ is

R^* :

	A	B	C
A	0.50	0.70	0.617
B	0.30	0.50	0.483
C	0.383	0.517	0.50

$\big(R^*(B, A) = 1 - 0.70, \ \ldots\big)$

Step 3: Decision rule. We compute the "score" of each alternative:

$s(\text{A}) = 0.50+0.70+0.617 = 1.817, \quad s(\text{B}) = 0.30+0.50+0.483 = 1.283, \quad s(\text{C}) = 0.383+0.517+0.50 = 1.400.$

Selecting the car(s) with maximal score yields the ranking

$$\text{Car A} \succ^* \text{Car C} \succ^* \text{Car B}.$$

Hence the group's preferred choice is **Car A**.

2 Main Results

In this section, we present the main contributions of this paper, namely the formal definitions and fundamental properties of HyperFuzzy Group Decision Making, SuperHyperFuzzy Group Decision Making, and their associated aggregation and decision rules.

2.1 HyperFuzzy Group Decision Making

HyperFuzzy Group Decision Making aggregates experts' hyperfuzzy preference sets to produce collective rankings or selections accounting for set-valued uncertainty.

Definition 2.1 (HyperFuzzy Group Decision Making)**.** Let $X = \{x_1, \ldots, x_n\}$ be a finite set of *alternatives* and $E = \{e_1, \ldots, e_m\}$ a finite set of *experts*. Denote by

$$\tilde{\mathcal{P}}([0,1]) \;=\; \{\, S \subseteq [0,1] \mid S \neq \emptyset \}$$

the family of all nonempty subsets of $[0,1]$. For each expert e_h, let

$$\tilde{R}_h : X \times X \;\longrightarrow\; \tilde{\mathcal{P}}([0,1])$$

be a *hyperfuzzy preference relation* satisfying for all $x_i, x_k \in X$:

1. reflexivity: $\tilde{R}_h(x_i, x_i) \subseteq [0,1]$ contains 0.5,
2. reciprocity: $\tilde{R}_h(x_k, x_i) = \{\, 1 - \alpha \mid \alpha \in \tilde{R}_h(x_i, x_k) \,\}$.

Define the *aggregation operator*

$$\tilde{\Phi} : \prod_{h=1}^{m} \tilde{\mathcal{P}}([0,1])^{X \times X} \;\longrightarrow\; \tilde{\mathcal{P}}([0,1])^{X \times X}$$

by

$$\tilde{R}^*(x_i, x_k) = \tilde{\Phi}\big(\tilde{R}_1(x_i, x_k), \ldots, \tilde{R}_m(x_i, x_k)\big) \;=\; \bigcup_{h=1}^{m} \tilde{R}_h(x_i, x_k),$$

for all $x_i, x_k \in X$. Finally, let D be a decision rule which, given the collective relation $\tilde{R}^*$, yields either

- a *ranking* via the hyperfuzzy outranking

$$x_i \succ^* x_k \iff \max \tilde{R}^*(x_i, x_k) \;\geq\; \max \tilde{R}^*(x_k, x_i),$$

- or a *selection* of optimal alternatives, for example by computing

$$s_i \;=\; \sum_{k=1}^{n} \max \tilde{R}^*(x_i, x_k), \quad \text{and choosing all } x_i \text{ with maximal } s_i.$$

The tuple

$$\big(X,\; E,\; \{\tilde{R}_h\}_{h=1}^m,\; \tilde{\Phi},\; D\big)$$

is called a *HyperFuzzy Group Decision Making* problem.

Example 2.2 (HyperFuzzy Group Decision Making: Smartphone Selection)**.** (cf. [60, 61]) Let

$$X = \{\text{Phone A, Phone B, Phone C}\}, \quad E = \{e_1, e_2, e_3\}.$$

Each expert e_h provides a hyperfuzzy preference relation $\tilde{R}_h : X \times X \to \tilde{\mathcal{P}}([0,1])$ with reflexivity $\tilde{R}_h(x,x) \supseteq \{0.5\}$ and reciprocity $\tilde{R}_h(y,x) = \{1 - \alpha : \alpha \in \tilde{R}_h(x,y)\}$.

Step 1: Expert hyperfuzzy preferences.

$$\begin{aligned}
&\tilde{R}_1(\mathrm{A,B}) = \{0.60, 0.70\}, && \tilde{R}_1(\mathrm{A,C}) = \{0.80\}, && \tilde{R}_1(\mathrm{B,C}) = \{0.50, 0.55\},\\
&\tilde{R}_2(\mathrm{A,B}) = \{0.65\}, && \tilde{R}_2(\mathrm{A,C}) = \{0.75, 0.85\}, && \tilde{R}_2(\mathrm{B,C}) = \{0.40, 0.60\},\\
&\tilde{R}_3(\mathrm{A,B}) = \{0.60, 0.65, 0.70\}, && \tilde{R}_3(\mathrm{A,C}) = \{0.80\}, && \tilde{R}_3(\mathrm{B,C}) = \{0.50, 0.60, 0.70\}.
\end{aligned}$$

Their reciprocals follow by $\tilde{R}_h(y,x) = \{1 - \alpha \mid \alpha \in \tilde{R}_h(x,y)\}$.

Step 2: Aggregation. Define $\tilde{R}^*(x, y) = \bigcup_{h=1}^{3} \tilde{R}_h(x, y)$. Then

$$\tilde{R}^*(\mathrm{A}, \mathrm{B}) = \{0.60, 0.65, 0.70\}, \quad \tilde{R}^*(\mathrm{A}, \mathrm{C}) = \{0.75, 0.80, 0.85\}, \quad \tilde{R}^*(\mathrm{B}, \mathrm{C}) = \{0.40, 0.50, 0.55, 0.60, 0.70\}.$$

Reciprocals:

$$\tilde{R}^*(\mathrm{B}, \mathrm{A}) = \{0.30, 0.35, 0.40\},$$
$$\tilde{R}^*(\mathrm{C}, \mathrm{A}) = \{0.15, 0.20, 0.25\},$$
$$\tilde{R}^*(\mathrm{C}, \mathrm{B}) = \{0.30, 0.40, 0.45, 0.50, 0.60\}.$$

Step 3: Decision rule (hyperfuzzy outranking). Compute maxima:

$$\max \tilde{R}^*(\mathrm{A}, \mathrm{B}) = 0.70,\ \max \tilde{R}^*(\mathrm{B}, \mathrm{A}) = 0.40$$
$$\Longrightarrow \quad \mathrm{A} \succ^* \mathrm{B},$$
$$\max \tilde{R}^*(\mathrm{A}, \mathrm{C}) = 0.85,\ \max \tilde{R}^*(\mathrm{C}, \mathrm{A}) = 0.25$$
$$\Longrightarrow \quad \mathrm{A} \succ^* \mathrm{C},$$
$$\max \tilde{R}^*(\mathrm{B}, \mathrm{C}) = 0.70,\ \max \tilde{R}^*(\mathrm{C}, \mathrm{B}) = 0.60$$
$$\Longrightarrow \quad \mathrm{B} \succ^* \mathrm{C}.$$

Thus the collective ranking is

$$\text{Phone A} \ \succ^* \ \text{Phone B} \ \succ^* \ \text{Phone C},$$

so the group's preferred choice is **Phone A**.

Theorem 2.3. *In a HyperFuzzy Group Decision Making problem as above:*

1. *The collective relation* $\tilde{R}^* \colon X \times X \to \tilde{\mathcal{P}}([0, 1])$ *is a* HyperFuzzy Set *on* $X \times X$.
2. *If each* $\tilde{R}_h(x_i, x_k)$ *is a singleton for all* h, i, k*, then the model reduces exactly to the classical Fuzzy Group Decision Making.*

Proof.
(1) By definition, $\tilde{R}^*(x_i, x_k)$ is the union of nonempty subsets of $[0, 1]$, hence itself a nonempty subset of $[0, 1]$. Thus $\tilde{R}^* \in \tilde{\mathcal{P}}([0, 1])^{X \times X}$, which is precisely the data of a HyperFuzzy Set on $X \times X$.
(2) If each expert's preference $\tilde{R}_h(x_i, x_k) = \{\mu_h(x_i, x_k)\}$ for some $\mu_h(x_i, x_k) \in [0, 1]$, then

$$\tilde{R}^*(x_i, x_k) = \bigcup_{h=1}^{m} \{\mu_h(x_i, x_k)\} = \{\max_{1 \le h \le m} \mu_h(x_i, x_k)\},$$

so the collective preference is the singleton-valued relation $\mu^*(x_i, x_k) = \max_h \mu_h(x_i, x_k)$. Hence $\mu^* \colon X \times X \to [0, 1]$ is an ordinary fuzzy preference relation, and one recovers exactly the classical Fuzzy Group Decision Making formulation. □

2.2 SuperHyperFuzzy Group Decision Making

SuperHyperFuzzy Group Decision Making extends aggregation by employing iterated powerset structures of alternatives and membership degrees to model hierarchical uncertainty.

Definition 2.4 ((m, n)-SuperHyperFuzzy Group Decision Making)**.** Let $X = \{x_1, \ldots, x_p\}$ be a finite set of *alternatives* and $E = \{e_1, \ldots, e_m\}$ a finite set of *experts*. Fix nonnegative integers m, n. Define the nonempty k-th powerset and its "tilde" extension by

$$\mathcal{P}_0^*(Y) = Y, \quad \mathcal{P}_k^*(Y) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(Y)\big) \setminus \{\emptyset\},$$
$$\tilde{\mathcal{P}}_k^*(Y) = \big\{ S \subseteq \mathcal{P}_k(Y) \mid S \neq \emptyset \big\}.$$

In particular $\mathcal{P}_m^*(X)$ is the domain of "super-alternatives" and $\tilde{\mathcal{P}}_n^*([0, 1])$ the family of allowed membership-degree sets at level n. A (m, n)-SuperHyperFuzzy Group Decision Making problem consists of:

- For each expert e_h, a *superhyperfuzzy preference relation*

$$\tilde{R}_h : \mathcal{P}^*_m(X) \times \mathcal{P}^*_m(X) \longrightarrow \tilde{\mathcal{P}}^*_n([0,1]),$$

  subject to

  1. (Reflexivity) $\kappa_n(\frac{1}{2}) \in \tilde{R}_h(A, A)$ for all $A \in \mathcal{P}^*_m(X)$,
  2. (Reciprocity) for each inner-level set $U \in \tilde{R}_h(A, B)$, $\mathrm{Comp}_n(U) \in \tilde{R}_h(B, A)$, where $\mathrm{Comp}_0(x) = 1 - x$ and $\mathrm{Comp}_{k+1}(S) = \{\mathrm{Comp}_k(s) \mid s \in S\}$.

- An *aggregation operator*

$$\tilde{\Phi} : \prod_{h=1}^{m} \Big(\tilde{\mathcal{P}}^*_n([0,1])\Big)^{\mathcal{P}^*_m(X)\times\mathcal{P}^*_m(X)} \longrightarrow \Big(\tilde{\mathcal{P}}^*_n([0,1])\Big)^{\mathcal{P}^*_m(X)\times\mathcal{P}^*_m(X)},$$

  defined pointwise by set–union:

$$\tilde{R}^*(A, B) = \tilde{\Phi}\big(\tilde{R}_1(A, B), \ldots, \tilde{R}_m(A, B)\big) = \bigcup_{h=1}^{m} \tilde{R}_h(A, B).$$

- A *decision rule D* which, given $\tilde{R}^*$, either
  - ranks “super-alternatives” by

$$A \succ^* B \iff \max^{(n)}\big(\tilde{R}^*(A, B)\big) \geq \max^{(n)}\big(\tilde{R}^*(B, A)\big),$$

    where

$$\overset{(0)}{\max}(x) = x$$

    ,

$$\overset{(k+1)}{\max}(S) = \max\{\overset{(k)}{\max}(s) \mid s \in S\}$$

    ,
  - or selects all A maximizing

$$s(A) = \sum_{B\in\mathcal{P}^*_m(X)} \max^{(n)}(\tilde{R}^*(A, B)).$$

The tuple

$$\big(X,\ E,\ \{\tilde{R}_h\}_{h=1}^{m},\ \tilde{\Phi},\ D\big)$$

is called a (m, n)-SuperHyperFuzzy Group Decision Making problem.

Example 2.5 (Real-world $(1, 2)$-SuperHyperFuzzy Group Decision Making: Supplier Evaluation)**.** Let

$$X = \{\text{Cost},\ \text{Quality},\ \text{Delivery}\}, \quad m = 1,\ n = 2, \quad E = \{e_1, e_2\}.$$

Then the “super-alternatives” domain is $\mathcal{P}^*_1(X)$, i.e. all nonempty subsets of X. In practice, the committee considers two major bundles:

$$A = \{\text{Cost}, \text{Quality}\},$$

$$B = \{\text{Quality}, \text{Delivery}\}.$$

Each expert e_h gives a superhyperfuzzy preference $\tilde{R}_h\colon \mathcal{P}^*_1(X) \times \mathcal{P}^*_1(X) \to \tilde{\mathcal{P}}^*_2([0,1])$.

Expert e_1:

$$\tilde{R}_1(A, B) = \Big\{\{\,0.70, 0.75\},\ \{\,0.85\}\Big\},$$

$$\tilde{R}_1(B, A) = \Big\{\{\,0.15, 0.25\},\ \{\,0.30\}\Big\},$$

with $\kappa_2(0.5) = \{\{0.5\}\} \in \tilde{R}_1(A, A)$.

Expert e_2:

$$\tilde{R}_2(A, B) = \Big\{\{\, 0.65\},\ \{\, 0.80, 0.90\}\Big\},$$

$$\tilde{R}_2(B, A) = \Big\{\{\, 0.10\},\ \{\, 0.20, 0.35\}\Big\},$$

with $\kappa_2(0.5) \in \tilde{R}_2(B, B)$.

Aggregating by union,

$$\tilde{R}^*(A, B) = \tilde{R}_1(A, B) \cup \tilde{R}_2(A, B) = \{\, \{0.65\}, \{0.70, 0.75\}, \{0.80, 0.90\}, \{0.85\}\},$$

$$\tilde{R}^*(B, A) = \{\, \{0.10\}, \{0.15, 0.25\}, \{0.20, 0.35\}, \{0.30\}\}.$$

Applying the hyperfuzzy outranking rule,

$$\overset{(2)}{\max}\big(\tilde{R}^*(A, B)\big) = \max\{\max\{0.65\}, \max\{0.70, 0.75\}, \max\{0.80, 0.90\}, \max\{0.85\}\} = 0.90,$$

$$\overset{(2)}{\max}\big(\tilde{R}^*(B, A)\big) = 0.35,$$

hence $A \succ^* B$. The group thus selects the “Cost + Quality” bundle A as the optimal supplier profile.

Example 2.6 (Real-world (2, 1)-SuperHyperFuzzy Group Decision Making: Urban Infrastructure Prioritization)**.** Let

$$X = \{\text{Transportation, Housing, GreenSpace, Industry}\}, \quad m = 2,\ n = 1, \quad E = \{e_1, e_2\}.$$

Then

$$\mathcal{P}_1^*(X) = \big\{\{\text{Transportation}\},\ \{\text{Housing}\},\ \{\text{GreenSpace}\},\ \{\text{Industry}\},$$
$$\{\text{Transportation,Housing}\},\ \ldots,\ \{\text{Transportation,Housing,GreenSpace,Industry}\}\big\},$$

and

$$\mathcal{P}_2^*(X) = \mathcal{P}\big(\mathcal{P}_1^*(X)\big) \setminus \{\emptyset\},$$

whose elements are *super-alternatives*, i.e. collections of nonempty subsets of X. In practice we focus on:

$$A = \big\{\{\text{Transportation,Housing}\},\ \{\text{Housing,GreenSpace}\}\big\},$$

$$B = \big\{\{\text{Transportation,Industry}\},\ \{\text{GreenSpace,Industry}\}\big\},$$

both in $\mathcal{P}_2^*(X)$.

Expert e_1:

$$\tilde{R}_1(A, B) = \big\{\{0.60, 0.75\},\ \{0.85\}\big\},$$

$$\tilde{R}_1(B, A) = \big\{\{0.15, 0.25\},\ \{0.30\}\big\},$$

with reflexivity $\{0.5\} \in \tilde{R}_1(A, A)$.

Expert e_2:

$$\tilde{R}_2(A, B) = \big\{\{0.65\},\ \{0.80, 0.90\}\big\},$$

$$\tilde{R}_2(B, A) = \big\{\{0.10\},\ \{0.20, 0.35\}\big\},$$

with $\{0.5\} \in \tilde{R}_2(B, B)$.

Aggregation: By union,

$$\tilde{R}^*(A, B) = \tilde{R}_1(A, B) \cup \tilde{R}_2(A, B) = \{\{0.60, 0.75\}, \{0.65\}, \{0.80, 0.90\}, \{0.85\}\},$$

$$\tilde{R}^*(B, A) = \{\{0.15, 0.25\}, \{0.30\}, \{0.10\}, \{0.20, 0.35\}\}.$$

Decision rule (hyperfuzzy outranking): Compute pointwise maxima:

$$\max \tilde{R}^*(A, B) = \max\{0.60, 0.75, 0.65, 0.80, 0.90, 0.85\} = 0.90, \quad \max \tilde{R}^*(B, A) = 0.35.$$

Since $0.90 > 0.35$, we have $A \succ^* B$. Thus the group’s optimal choice is the super-alternative A, prioritizing the “Transportation & Housing” and “Housing & GreenSpace” bundles.

Theorem 2.7. *Let $(X, E, \{\tilde{R}_h\}, \tilde{\Phi}, D)$ be a (m, n)-SuperHyperFuzzy Group Decision Making problem as above. Then:*

1. *The aggregated relation $\tilde{R}^* : \mathcal{P}^*_m(X) \times \mathcal{P}^*_m(X) \to \tilde{\mathcal{P}}^*_n([0, 1])$ is precisely an (m, n)-SuperHyperFuzzy Set on $\mathcal{P}^*_m(X)$.*
2. *If $m = n = 0$, then $\mathcal{P}^*_0(X) = X$ and $\tilde{\mathcal{P}}^*_0([0, 1]) = [0, 1]$, so the model reduces to classical Fuzzy Group Decision Making.*
3. *If $m = 0$ and $n > 0$, then it reduces to HyperFuzzy Group Decision Making.*

Proof. (1) By construction, for each (A, B), $\tilde{R}^*(A, B)$ is a nonempty union of elements in $\mathcal{P}_n([0, 1])$, so $\tilde{R}^*(A, B) \in \tilde{\mathcal{P}}^*_n([0, 1])$. Hence $\tilde{R}^*$ is exactly an (m, n)-SuperHyperFuzzy Set on $\mathcal{P}^*_m(X)$.

(2) If $m = n = 0$, then by definition $\mathcal{P}^*_0(X) = X$ and $\tilde{\mathcal{P}}^*_0([0, 1]) = [0, 1]$. Each expert relation $\tilde{R}_h$ becomes a function $X \times X \to [0, 1]$, and aggregation by union becomes pointwise maximum. This coincides with the classical fuzzy-preference aggregation.

(3) If $m = 0$ but $n > 0$, then $\mathcal{P}^*_0(X) = X$ while $\tilde{\mathcal{P}}^*_n([0, 1])$ remains nonempty subsets of $\mathcal{P}_n([0, 1])$. Thus one recovers exactly the HyperFuzzy Group Decision Making formulation. □

Theorem 2.8 (Aggregation Closure)**.** *Let $(X, E, \{\tilde{R}_h\}_{h=1}^m, \tilde{\Phi}, D)$ be a (m, n)-SuperHyperFuzzy Group Decision Making problem, and define the collective relation*

$$\tilde{R}^*(A, B) = \tilde{\Phi}\big(\tilde{R}_1(A, B), \dots, \tilde{R}_m(A, B)\big) = \bigcup_{h=1}^{m} \tilde{R}_h(A, B),$$

for all $A, B \in \mathcal{P}^*_m(X)$. Then $\tilde{R}^*$ is again a valid superhyperfuzzy preference relation, i.e.

$$\tilde{R}^* : \mathcal{P}^*_m(X) \times \mathcal{P}^*_m(X) \longrightarrow \tilde{\mathcal{P}}^*_n([0, 1]),$$

satisfying reflexivity and reciprocity.

Proof. By definition each $\tilde{R}_h(A, B)$ is a nonempty subset of $\mathcal{P}_n([0, 1])$. A union of nonempty subsets is nonempty, so $\tilde{R}^*(A, B) \in \tilde{\mathcal{P}}^*_n([0, 1])$. For reflexivity, since $\kappa_n(\frac{1}{2}) \in \tilde{R}_h(A, A)$ for each h, it follows $\kappa_n(\frac{1}{2}) \in \bigcup_{h=1}^m \tilde{R}_h(A, A) = \tilde{R}^*(A, A)$. For reciprocity, if $U \in \tilde{R}^*(A, B)$ then $U \in \tilde{R}_{h_0}(A, B)$ for some h_0, hence by assumption $\mathrm{Comp}_n(U) \in \tilde{R}_{h_0}(B, A) \subseteq \tilde{R}^*(B, A)$. This shows $\tilde{R}^*$ satisfies both axioms, proving the claim. □

Theorem 2.9 (Consensus Idempotence)**.** *If all experts agree pointwise, i.e. $\tilde{R}_1 = \tilde{R}_2 = \cdots = \tilde{R}_m =: R$, then the aggregated relation coincides with R:*

$$\tilde{R}^*(A, B) = \bigcup_{h=1}^{m} R(A, B) = R(A, B) \quad \forall A, B \in \mathcal{P}^*_m(X).$$

Proof. Since each $\tilde{R}_h(A, B) = R(A, B)$, the union $\bigcup_{h=1}^m R(A, B)$ simply returns the same set $R(A, B)$, establishing idempotence under unanimous expert opinion. □

Theorem 2.10 (Score Monotonicity)**.** *Let $(X, E, \{\tilde{R}_h\}, \tilde{\Phi}, D)$ be a (m, n)-SuperHyperFuzzy GDM problem with aggregated relation $\tilde{R}^*(A, B) = \bigcup_{h=1}^m \tilde{R}_h(A, B)$. Define the "score" of a super-alternative A by*

$$s(A) = \sum_{B \in \mathcal{P}^*_m(X)} \max{}^{(n)}\big(\tilde{R}^*(A, B)\big).$$

If for some expert index t and some pair (A_0, B_0) we replace $\tilde{R}_t(A_0, B_0)$ by a strictly larger nonempty set $\tilde{R}'_t(A_0, B_0) \supsetneq \tilde{R}_t(A_0, B_0)$, then the new aggregated relation $\tilde{R}^{*\prime}$ satisfies

$$s'(A_0) \ \geq \ s(A_0),$$

and $s'(A) = s(A)$ for all $A \neq A_0$.

Proof. Only the union at (A_0, B_0) changes:

$$\tilde{R}^{*'}(A_0, B_0) = \Big(\bigcup_{h \neq t} \tilde{R}_h(A_0, B_0)\Big) \cup \tilde{R}'_t(A_0, B_0) \supseteq \tilde{R}^*(A_0, B_0).$$

Hence $\max^{(n)}\big(\tilde{R}^{*'}(A_0, B_0)\big) \geq \max^{(n)}\big(\tilde{R}^*(A_0, B_0)\big)$, while for any other pair $(A, B) \neq (A_0, B_0)$ the aggregated set is unchanged, so its maximum is unchanged. Summing over B yields $s'(A_0) \geq s(A_0)$ and $s'(A) = s(A)$ for $A \neq A_0$. □

Theorem 2.11 (Redundancy of Dominated Expert)**.** *In the same setting, suppose there is an expert index t such that for every pair (A, B)*

$$\tilde{R}_t(A, B) \subseteq \bigcup_{h \neq t} \tilde{R}_h(A, B).$$

Then removing expert t does not change the aggregated relation:

$$\bigcup_{h=1}^{m} \tilde{R}_h(A, B) = \bigcup_{h \neq t} \tilde{R}_h(A, B),$$

$$\forall\, A, B \in \mathcal{P}^*_m(X).$$

Proof. For each (A, B),

$$\bigcup_{h=1}^{m} \tilde{R}_h(A, B) = \Big(\bigcup_{h \neq t} \tilde{R}_h(A, B)\Big) \cup \tilde{R}_t(A, B)$$
$$= \bigcup_{h \neq t} \tilde{R}_h(A, B),$$

since by hypothesis $\tilde{R}_t(A, B)$ is already contained in the union over $h \neq t$. Thus the aggregate is unchanged when t is omitted. □

Theorem 2.12 (Commutativity and Associativity of Aggregation)**.** *The union-based aggregation operator $\tilde{\Phi}\big(\tilde{R}_1, \ldots, \tilde{R}_m\big) = \bigcup_{h=1}^{m} \tilde{R}_h$ is both commutative and associative in the expert index: for any reordering or grouping of the $\tilde{R}_h$, the result of $\tilde{R}^*$ is identical.*

Proof. Union of sets is well-known to satisfy $A \cup B = B \cup A$ (commutativity) and $(A \cup B) \cup C = A \cup (B \cup C)$ (associativity). Applying these identities repeatedly across the family $\{\tilde{R}_h(A, B)\}_{h=1}^{m}$ yields the desired result. □

3 Conclusion

In this paper, we introduced the Hyperfuzzy and (m, n)-SuperHyperfuzzy Group Decision Making frameworks. Future work will explore group decision making approaches based on Graphs [62, 63], Directed Graph [64–66], Hypergraphs [67–70], Plithogenic Set [19, 20, 71], and SuperHyperGraphs [72–76], and will include experimental evaluations using various algorithms and datasets.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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